Classical Mechanics

Chapter 9. Canonical Transformations.

Homework

Problem 5A. **(10 points)**

Construct from the first principles the Hamiltonian for a 1D harmonic oscillator of mass ***m*** and spring constant ***k***. Determine the value of the constant ***C*** such that the following equations define a canonical transformation from the old variables (q, p) to the new variables (Q, P):

$$Q = C(p + im\omega q)$$
$$P = C(p - im\omega q)$$

Where $\omega = \sqrt{k/m}$. What is the generating function for this transformation? Find Hamilton's equations of motion for the new variables and integrate them. Hence find the solution to the original problem.

Problem 5B. **(10 points)**

The Hamiltonian for a particle moving in a vertical uniform gravitational field ***g*** is

$$H = \frac{p^2}{2m} + mgq$$

where q is the altitude above the ground. We want to find any canonical transformation from old variables *(q,p)* to new variables *(Q,P)* which provides a cyclic coordinate. To do this, define new variables as

$$Q = bp \qquad P = aH$$

*where **a, b** are constants.*

a) Determine any combination of constants a and b, which provides a canonical transformation.

b) Find the type 1 generating function, F_1 (q, Q)

c) Use the relation $F_2(q, P) = F_1 + PQ$ to find the type 2 generating function and check your result by showing that F_2 indeed generates the same transformation

d) Find the new Hamiltonian K for the new canonical variables Q, P.
Are there any cyclic variables?

e) *Solve Hamilton equations for the new canonical variables*

Find the original variables q,p as a function of time

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Problem A. **(10 points)**

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$T = \frac{1}{2} m\dot{q}^2$; $V = \frac{1}{2} k q^2 = \left| \omega^2 = \frac{k}{m} \right| = \frac{1}{2} m\omega^2 q^2$; $L = T - V = \frac{m\dot{q}^2}{2} - \frac{m\omega^2 q^2}{2}$

- quadratic f-n of $\dot{q}$ - f-n of q only $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \Rightarrow \dot{q} = P/m$

so, the Hamiltonian is: $(L \neq f(t))$

$$\tilde{\mathcal{H}} = T + V = \frac{1}{2} m\dot{q}^2 + \frac{1}{2} m\omega^2 q^2$$

$$(1) \quad \boxed{\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}}$$

A canonical transformation must satisfy: $[Q,P] = 1$
$[Q,Q] = 0 = [P,P]$

From these we can get C:

$[Q,Q] = [P,P] = 0$ (nothing usefull here)

$$[Q,P] = 1 = \frac{\partial Q}{\partial q}\cdot\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\cdot\frac{\partial P}{\partial q} = \left[\left(C\cdot(im\omega)\cdot C\right) - \left(C^2(-im\omega)\right)\right] =$$

$$= 2 C^2 i m\omega$$

$$\Rightarrow \boxed{C = \frac{1}{\sqrt{2im\omega}}}$$ with this C the transform. $\{q,p\} \to \{Q,P\}$ is canonical.

Generating f-n?

Let's find $F_2(q, P)$

(2) $p = \frac{\partial F_2}{\partial q}$; (3) $Q = \frac{\partial F_2}{\partial P}$

so we need to get $p(q,P)$, $Q(q,P)$ from the transform. eq-ns to use them in (2) and (3)

$$\begin{cases} Q = c(p + im\omega q) \\ P = c(p - im\omega q) \end{cases} \Rightarrow \begin{cases} Q = c\cdot\left(\frac{P}{c} + im\omega q + im\omega q\right) = P + i\cdot c\cdot m\omega q\cdot 2 \\ p = \frac{P}{c} + im\omega q \end{cases}$$

$$(2) \Rightarrow \frac{\partial F_2(q,P)}{\partial q} = \frac{P}{c} + im\omega q$$

$$F_2(q,P) = \frac{P}{c}\cdot q + im\omega\frac{q^2}{2} + f(P)$$

$$(3) \Rightarrow Q(q,P) = \frac{\partial F_2}{\partial P} = \frac{q}{c} + \frac{\partial f(P)}{\partial P} = P + i\cdot c\cdot m\omega q\cdot 2$$

$$\text{so } \frac{\partial f(P)}{\partial P} = P \Rightarrow f(P) = \frac{P^2}{2}$$

so, finally

$$\boxed{F_2(q,P) = \frac{P}{c}q + i\cdot m\omega\cdot\frac{q^2}{2} + \frac{P^2}{2}} \qquad F_1 = \frac{qQ}{c} - i\frac{1}{2}m\omega q^2 + \frac{Q^2}{2}$$

now, the new Hamiltonian is

$$\tilde{K}(Q,P) = H(q,p) + \cancel{\frac{\partial F_2}{\partial t}}^{\,0} = H(q,p) = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$$

$$\begin{cases} Q = c(p + im\omega q) \\ P = c(p - im\omega q) \end{cases} \Rightarrow \begin{cases} p = \frac{P+Q}{2c} \\ q = \frac{Q-P}{2c\,im\omega} \end{cases}$$

$$K(Q,P) = \frac{1}{2m}\left(\frac{P+Q}{2c}\right)^2 + \frac{m\omega^2}{2}\cdot\left(\frac{Q-P}{2c\cdot im\omega}\right)^2 = \frac{(P+Q)^2 - (Q-P)^2}{8m\cdot c^2} =$$

$$= \frac{4P\cdot Q}{8mc^2} = \frac{4P\cdot Q}{8m}\cdot 2im\omega = i\cdot\omega\cdot P\cdot Q \Rightarrow \boxed{K(Q,P) = i\omega P\cdot Q}$$

Eq-ns of motion.

$$\dot{Q} = \frac{\partial K}{\partial P} = i\cdot\omega\cdot Q \Rightarrow Q = Q_0\cdot e^{i\omega t}$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = -i\cdot\omega\cdot P \Rightarrow P = P_0\cdot e^{-i\omega t}$$

now we need to go back to the original gener. coordin.

$$\therefore\ p = \frac{P+Q}{2\cdot C} = \left(\frac{P_0\cdot e^{-i\omega t} + Q_0\cdot e^{i\omega t}}{2}\right)\cdot\sqrt{2im\omega} = \sqrt{\frac{im\omega}{2}}\cdot\left(Q_0 e^{i\omega t} + P_0\cdot e^{-i\omega t}\right)$$

$$\therefore\ q = \frac{Q-P}{2im\omega\cdot C} = \frac{1}{\sqrt{2im\omega}}\left(Q_0 e^{i\omega t} - P_0 e^{-i\omega t}\right)$$

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d) Find the new Hamiltonian K for the new canonical variables Q, P. Is there any cyclic variables?

e) Solve Hamilton equations for the new canonical variables

f) Find the original variables q,p as a function of time

a) The transformation is canonical when these conditions are satisfied

$$[Q,Q] = [P,P] = 0$$

$$[Q,P] = 1, \text{ so}$$

$$[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = -b \cdot amg = 1$$

where we use
$$\begin{cases} Q = b \cdot p \\ P = a\left(\frac{p^2}{2m} + mgq\right) \end{cases}$$

We have some freedom and let's choose

$a = 1$; $b = -\frac{1}{mg}$, so our transform. eq-ns

$$(1) \begin{cases} P = \frac{p^2}{2m} + mgq & \text{(the new momentum is our Hamiltonian!)} \\ Q = -\frac{p}{mg} \end{cases}$$

b) $F_1(q, Q)$ $\quad p = \frac{\partial F_1}{\partial q} \quad P = -\frac{\partial F_1}{\partial Q}$

from (1) we can get

$$(2) \begin{cases} p = -mgQ \\ P = \frac{mg^2Q^2}{2} + mgq \end{cases} \Rightarrow \begin{cases} p = -mgQ \\ q = \frac{1}{mg} \cdot \left(P - \frac{mg^2Q^2}{2}\right) \end{cases}$$

$$p = \frac{\partial F_1}{\partial q} = -mgQ \Rightarrow F_1(q, Q) = -mgQ \cdot q + f(Q)$$

$$P = -\frac{\partial F_1}{\partial Q} = mgq - \frac{\partial f(Q)}{\partial Q} = \| P \text{ from (2)} \| = \frac{mg^2Q^2}{2} + mgq$$

$$\text{so} \quad \frac{\partial f(Q)}{\partial Q} = \frac{d f(Q)}{dQ} = -\frac{mg^2Q^2}{2}$$

$$f(Q) = -\frac{1}{6} mg^2 Q^3$$

$$\boxed{F_1(q, Q) = -mgQq - \frac{1}{6} mg^2 Q^3}$$

c) $F_2(q, P) = F_1(q, Q) + PQ$

To remove Q, we need $Q = Q(q, P)$

From (2) $\Rightarrow$ $Q = \sqrt{\frac{2}{mg^2} \cdot (P - mgq)}$

$$F_2 = -mgQq - \frac{1}{6} mg^2 Q^3 + PQ = Q \cdot \left[-mgq - \frac{1}{\cancel{6}_3} mg^2 \left(\frac{\cancel{2}}{mg^2} (P - mgq)\right) + P\right] =$$

$$= Q\left[-mgq - \frac{P}{3} + \frac{mgq}{3} + P\right] = \frac{2}{3} (P - mgq) \sqrt{\frac{2}{mg^2} \cdot (P - mgq)}$$

$$\boxed{F_2(q, P) = \frac{2\sqrt{2}}{3 \cdot \sqrt{mg^2}} \cdot (P - mgq)^{3/2}}$$

$$\begin{cases} p = \dfrac{\partial F_2}{\partial q} = \dfrac{\cancel{2}\sqrt{2}}{3\cdot\sqrt{mg^2}}\cdot\dfrac{3}{\cancel{2}}\cdot(P-mgq)^{1/2}\cdot(-m\cancel{g}) = -\sqrt{2m}\,(P-mgq)^{1/2} \\ Q = \dfrac{\partial F_2}{\partial P} = \sqrt{\dfrac{2}{mg^2}}\cdot(P-mgq)^{1/2} \end{cases}$$

so $\begin{cases} p = -m\cdot g\cdot Q \quad \text{and from the 2nd eq-n} \\ q = \dfrac{1}{mg}\cdot\left(P - \dfrac{mQ^2g^2}{2}\right) \end{cases}$

which are exactly what we had in (2). The same transformation eq-ns.

d) New Hamiltonian, K -?

$$\tilde{K} = \mathcal{H} + \cancel{\frac{\partial F_2}{\partial t}}^{\,0} = \frac{p^2}{2m} + mgq \Rightarrow$$

$$K = \frac{(-mgQ)^2}{2m} + \cancel{mg}\cdot\frac{1}{\cancel{mg}}\cdot\left(P - \frac{mQ^2g^2}{2}\right) =$$

$$= \cancel{\frac{mg^2Q^2}{2}} + P - \cancel{\frac{mg^2Q^2}{2}} = P, \text{ so } \boxed{K(Q,P) = P}$$

so, Q is cyclic.

e) $\left.\begin{array}{l} \dot{Q} = \dfrac{\partial K}{\partial P} = 1 \\ \dot{P} = -\dfrac{\partial K}{\partial Q} = 0 \end{array}\right\}$ Awesome. $\ddot{\smile}$ $\Rightarrow \boxed{\begin{array}{l} Q(t) = t + \beta' \\ P = \alpha \end{array}}$

α, β' – const of integr

d) let's go back to (q,p). Use the transf. eq-ns (2)

$$p = -mg\cdot Q = -mgt - mg\cdot\beta' = \| -mg\beta' = \beta \| = -mgt + \beta$$

$$q = \frac{1}{mg}\left(P - \frac{mg^2Q^2}{2}\right) = \frac{1}{mg}\left(\alpha - \frac{mg^2\cdot(t+\beta')^2}{2}\right) =$$

$$= \frac{\alpha}{mg} - \frac{g}{2}\cdot(t^2 + 2t\beta' + \beta'^2)$$

$$q = -\frac{g}{2}\cdot t^2 - \left(\frac{\cancel{2}\cdot\beta' g}{\cancel{2}}\right)\cdot t - \frac{g\cdot\beta'^2}{2} + \frac{\alpha}{mg}$$

so

$$\boxed{\begin{cases} q = \left(\dfrac{\alpha}{mg} - \dfrac{g\beta'^2}{2}\right) - (g\beta')t - \dfrac{g}{2}\cdot t^2 \\ p = \beta - mgt \end{cases}}$$

P.S. By redefining constants, we can get a typical kinematic eq-ns $(q = y)$

$$\begin{cases} y = y_0 + \dfrac{p_0}{m}\cdot t + \dfrac{1}{2}\cdot g\cdot t^2 \\ p = p_0 + mgt \end{cases}$$