9.2 A donut-shaped space station (outer radius \( R \)) arranges for artificial gravity by spinning on the axis of the donut with angular velocity \( \omega \). Sketch the forces on, and accelerations of, an astronaut standing in the station (a) as seen from an inertial frame outside the station and (b) as seen in the astronaut’s personal rest frame (which has a centripetal acceleration \( A = \omega^2 R \) as seen in the inertial frame). What angular velocity is needed if \( R = 40 \) meters and the apparent gravity is to equal the usual value of about 10 m/s\(^2\)? (c) What is the percentage difference between the perceived \( g \) at a six-foot astronaut’s feet (\( R = 40 \) m) and at his head (\( R = 38 \) m)?

\[ A = \omega^2 R \]

**Acentripetal force makes the astronaut follow the circular path and it is equal to the normal force \( N \). \( N = F_{op} \). So, the astronaut experiences a centripetal acceleration \( (\omega^2 R) \) provided by the normal force. (seen by an inertial observer)

**C**

\[ \delta(\%) = \frac{\text{foot} - \text{head}}{\text{foot}} \times 100\% = \frac{40 - 38}{40} \times 100\% = 5\% \]

\[ \omega^2 \frac{R_{\text{foot}} - R_{\text{head}}}{R_{\text{foot}}} \]

\[ \omega^2 \frac{40 - 38}{40} = 0.5 \text{ rad/s} = 4.8 \text{ rpm.} \]
9.26 In Section 9.8, we used a method of successive approximations to find the orbit of an object that is dropped from rest, correct to first order in the earth's angular velocity \( \Omega \). Show in the same way that if an object is thrown with initial velocity \( \mathbf{v}_0 \) from a point \( O \) on the earth’s surface at colatitude \( \theta \), then to first order in \( \Omega \) its orbit is

\[
\begin{align*}
  x &= v_{0x}t + \Omega (v_{0x} \cos \theta - v_{0x} \sin \theta) t^2 + \frac{1}{2} \Omega g t^2 \sin \theta \\
  y &= v_{0y}t - \Omega (v_{0x} \cos \theta) t^2 \\
  z &= v_{0z}t - \frac{1}{2} gt^2 + \Omega (v_{0x} \sin \theta) t^2.
\end{align*}
\]  

(9.73)

First solve the equations of motion (9.53) in zeroth order, that is, ignoring \( \Omega \) entirely. Substitute your zeroth-order solution for \( \dot{x} \), \( \dot{y} \), and \( \dot{z} \) into the right side of equations (9.53) and integrate to give the next approximation. Assume that \( v_0 \) is small enough that air resistance is negligible and that \( g \) is a constant throughout the flight.

The eqns of motion on the earth surface

\[
\ddot{\mathbf{r}} = m \ddot{\mathbf{g}}_0 + \mathbf{F}_{\text{cor}} + \mathbf{F}_{\text{eff}}, \quad \mathbf{F}_{\text{eff}} = m \ddot{\mathbf{g}}_0 + \mathbf{F}_{\text{cor}} = m \ddot{\mathbf{g}}
\]

\[
\ddot{\mathbf{r}} = \ddot{\mathbf{g}} + \frac{\mathbf{F}_{\text{cor}}}{m} = \ddot{\mathbf{g}} + \frac{2m(\ddot{\mathbf{r}} \times \dot{\mathbf{r}})}{m} = \ddot{\mathbf{g}} + 2 \dddot{\mathbf{r}} \times \dot{\mathbf{r}}
\]

Since it depends on \( \dddot{\mathbf{r}} \), \( \dot{\mathbf{r}} \) we can move the coord system from the earth center to the surface

\( \hat{x} \) (east), \( \hat{y} \) (north), \( \hat{z} \) (up)

In this coord system

\[
\mathbf{V}_0 = (V_{0x}, V_{0y}, V_{0z})
\]

\[
\mathbf{a} = (a_x, a_y, a_z) = \omega \times (\omega \times \mathbf{r})
\]

\[
\ddot{\mathbf{g}} = (0, 0, -g)
\]

\[
\dddot{\mathbf{r}} = \ddot{\mathbf{g}} + 2 \dddot{\mathbf{r}} \times \dot{\mathbf{r}}
\]

Solve it using perturbation analysis.

Power series in a small parameter \( \epsilon \):

\[
\dddot{\mathbf{r}}(t) = \dddot{\mathbf{r}}_0(t) + \epsilon \dddot{\mathbf{r}}_1(t) + \epsilon^2 \dddot{\mathbf{r}}_2(t) + \ldots
\]

Put it in the DE:

\[
\dddot{\mathbf{r}}_0 + \epsilon \dddot{\mathbf{r}}_1 + \epsilon^2 \dddot{\mathbf{r}}_2 + \ldots = \dddot{\mathbf{g}} + \epsilon \left( \dddot{\mathbf{r}}_0 + \epsilon \dddot{\mathbf{r}}_1 + \epsilon^2 \dddot{\mathbf{r}}_2 + \ldots \right) \times \dot{\mathbf{r}}
\]

The corresponding terms must be equal, so
(1) \( \frac{d^2 \vec{r}_0}{dt^2} = \vec{g} \)

(2) \( \frac{d^2 \vec{r}_i}{dt^2} = 2 \frac{d \vec{r}_0}{dt} \times \hat{\vec{n}}, \quad \hat{\vec{n}} = \frac{\vec{n}}{|\vec{n}|} \)

Solve (1):

\[ \vec{r}_0 = \vec{g}t + \vec{v}_0 \Rightarrow \vec{r}_0(t) = \vec{g} \frac{t^2}{2} + \vec{v}_0 t + \vec{r}_0(0) \]

Put it into (2),

\[ \frac{d^2 \vec{r}_i}{dt^2} = 2 \left( \vec{g}t + \vec{v}_0 \right) \times \vec{r}_i \]

\[ \vec{r}_i(t) = (\vec{g} \times \vec{r}_i)t^2 + 2(\vec{v}_0 \times \vec{r}_i)t \]

\[ \vec{r}_i(t) = (\vec{g} \times \vec{r}_i) \frac{t^3}{3} + (\vec{v}_0 \times \vec{r}_i)t^2 \], so the final solution is

\[ \vec{r}_i(t) = \vec{r}_0 + \vec{r}_i = \vec{g} \frac{t^2}{2} + \vec{v}_0 t + \vec{r}_0 + (\vec{g} \times \vec{r}_i) \frac{t^3}{3} + (\vec{v}_0 \times \vec{r}_i)t^2 \]

The eqn that can be used for ANY initial conditions, let's apply to ours.

\[ \vec{r}_i(0) = \vec{0} \quad ; \quad \vec{v}_i = (v_{i0x}, v_{i0y}, v_{i0z}) \]

\[ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & -g \\ 0 & l \cdot \sin \theta & l \cdot \cos \theta \end{vmatrix} = (g \cdot \sin \theta, 0, 0) \]

\[ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_{i0x} & v_{i0y} & v_{i0z} \\ 0 & l \cdot \sin \theta & l \cdot \cos \theta \end{vmatrix} = \hat{i} (v_{i0y} \cdot \cos \theta - v_{i0z} \cdot \sin \theta) - \hat{j} (v_{i0z} \cdot \cos \theta + v_{i0y} \cdot \sin \theta) + \hat{k} (v_{i0z} \cdot \sin \theta - v_{i0y} \cdot \cos \theta) \]

So now we can decompose \( \vec{r}_i(t) \) into \( \hat{x}, \hat{y}, \hat{z} \):

\[ \vec{r}_i(t) = -\frac{g \cdot t^2}{2} \hat{z} + v_{i0x} \cdot t \hat{x} + v_{i0y} \cdot t \hat{j} + v_{i0z} \cdot \hat{z} + g \cdot \sin \theta \cdot \hat{x} \cdot \frac{t^3}{3} + \]

\[ + \hat{x} \cdot t^2 (v_{i0y} \cdot \cos \theta - v_{i0z} \cdot \sin \theta) - \hat{y} \cdot t (v_{i0z} \cdot \cos \theta + v_{i0y} \cdot \sin \theta) + \hat{z} \cdot t^2 v_{i0z} \cdot \sin \theta \]

\[ \begin{cases} \[x(t) = v_{i0x} \cdot t + \frac{1}{2} (v_{i0y} \cdot \cos \theta - v_{i0z} \cdot \sin \theta) \cdot t^2 + \frac{1}{3} g \cdot \sin \theta \cdot t^3 \cdot \sin \theta \] \\
[y(t) = v_{i0y} \cdot t - \frac{1}{2} (v_{i0z} \cdot \sin \theta + v_{i0y} \cdot \cos \theta) \cdot t^2] \\
[z(t) = v_{i0z} \cdot t - \frac{g \cdot t^2}{2} + \frac{l \cdot (v_{i0z} \cdot \sin \theta)}{2} \cdot t^2] \end{cases} \]
9.28 ** Use the result (9.73) of Problem 9.26 to do the following: A naval gun shoots a shell at colatitude $\theta$ in a direction that is $\alpha$ above the horizontal and due east, with muzzle speed $v_0$. (a) Ignoring the earth's rotation (and air resistance), find how long $t$ the shell would be in the air and how far away $R$ it would land. If $v_0 = 500$ m/s and $\alpha = 20^\circ$, what are $t$ and $R$? (b) A naval gunner spots an enemy ship due east at the range $R$ of part (a) and, forgetting about the Coriolis effect, aims his gun exactly as in part (a). Find by how far north or south, and in which direction, the shell will miss the target, in terms of $\Omega$, $v_0$, $\alpha$, $\theta$, and $g$. (It will also miss in the east–west direction but this is perhaps less critical.) If the incident occurs at latitude 50° north ($\theta = 40^\circ$), what is this distance? What if the latitude is 50° south? (This problem is a serious issue in long-range gunnery: In a battle near the Falkland Islands in World War I, the British navy consistently missed German ships by many tens of yards because they apparently forgot that the Coriolis effect in the southern hemisphere is opposite to that in the north.)

\[ \vec{F} = m\vec{a} = m\vec{g}/(\hat{z}) \]

- **Problem**: \[ x'' = 0 \]
  - $x = At + B$ apply the i.c.
  - $x(t) = V_{ax}t + x_0$
  - $x(t) = V_0 \cdot \cos \alpha \cdot t = V_{ax}t$

\[ z = 0 \Rightarrow 0 = t \left( -\frac{gt^2}{2} + V_0 \sin \theta \right) \]

Range? \[ x(t_2) = V_0 \cdot \cos \alpha \left( \frac{2V_0 \cdot \sin \theta}{g} \right) = \frac{2V_0^2 \cdot \sin \theta \cdot \cos \theta}{g} = \frac{2V_0^2 \cdot \sin 2\theta}{g} = R \]

if $v_0 = 500$ m/s; $\alpha = 20^\circ$, then
\[ t_2 = 34.9 \sec \; \text{and} \; R = 16.4 \text{km}. \]
Here we need to apply eq-us (9.73) from Problem 9.26 to our initial conditions:

\[ \vec{V}_0 = (V_0 \cdot \cos \theta, 0, V_0 \cdot \sin \theta) \]

\[ \vec{V} = (0, R \cdot \sin \theta, R \cdot \cos \theta) \]

\[ \vec{g} = (0, 0, -g) \]

\[ x(t) = V_0 \cos \theta \cdot t + \frac{1}{2} (0.5g - V_0 \cdot \sin \theta \cdot \sin \theta) t^2 + \frac{1}{2} g t^2 \sin \theta \]

\[ y(t) = 0.5 - 2V_0 \cos \theta \cdot \cos \theta \cdot t^2 \]

\[ z(t) = \frac{V_0 \cdot \sin \theta}{g} \cdot t^2 + \frac{1}{2} V_0 \cdot \cos \theta \cdot \sin \theta \cdot t^2 \]

The direction determines north/south deflection.

If \( \theta = 40^\circ \Rightarrow y(t) = -(7.3 \cdot 10^{-5}) \cdot (500 \text{ m/s}) \cdot \cos 20^\circ \cdot \cos 40^\circ \cdot t^2 \]

Oops! We still need to find \( t \). We can get it from \( z(t) \) eq-u, where \( z(t) = 0 \), so

\[ t = \frac{V_0 \cdot \sin \theta}{g} - \frac{1}{2} V_0 \cdot \cos \theta \cdot \sin \theta \cdot \cos \theta \]

\[ = \frac{V_0 \cdot \sin \theta}{g - 2 \cdot V_0 \cdot \cos \theta \cdot \sin \theta} \approx 35.5^s \]

It's slightly more than 34.5 s from part a. A very small correction.

So, now we can go back to our calculations of \( y(t) \)

\[ y(t) = -32.18 \text{ m} \] means deflection to the south.

If the latitude is \( 80^\circ \) south? It means \( \theta = 90^\circ + 80^\circ = 180^\circ \)

\( \cos 80^\circ = -0.173 \), so we'll get extra miles, which means that the shell will be deflected to the north in the southern hemisphere by 32.18 m.