Lecture 20

Chapter 11

Conservation of Angular momentum

Course website:
http://faculty.uml.edu/Andriy_Danylov/Teaching/PhysicsI

Lecture Capture:
http://echo360.uml.edu/danylov2013/physics1spring.html
Exam III Info

Exam III Friday April 25, 9:00-9:50am, OH 150.
Exam III covers Chapters 9-11
Same format as Exam II
Prior Examples of Exam III posted

Ch. 9: Linear Momentum (no section 10)
Ch. 10: Rotational Motion (no section 10)
Ch. 11: Angular Momentum; General Rotation (no sections 7-9)

Exam Review Session TBA
Outline

Chapter 11

- Conservation of Angular Momentum
- Kepler’s 2nd Law revisited
Two definitions of Angular Momentum

Single particle

\[ \vec{L} = \vec{r} \times \vec{p} \]

Rigid symmetrical body

\[ \vec{L} = I \vec{\omega} \]

L is perpendicular to the “plane of motion” created by r and p
Rotational N.2\textsuperscript{nd} law

Torque causes angular momentum to change

\[ \frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}} \]

Total angular momentum of a system

Net Torque of a system

It is similar to translational N. 2\textsuperscript{nd} law

Translational N.2\textsuperscript{nd} law

\[ \vec{F} = m\vec{a} \]
\[ \vec{F} = \frac{d\vec{p}}{dt} \]

Rotational N.2\textsuperscript{nd} law

\[ \vec{\tau} = I\vec{\alpha} \]
\[ \vec{\tau} = \frac{d\vec{L}}{dt} \]
Conservation of Angular Momentum

Angular momentum is an important concept because, under certain conditions, it is conserved.

\[
\text{Rotational N.2nd law} \quad \frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}}
\]

If the net external torque on an object is zero, then the total angular momentum is conserved.

\[
\text{If } \vec{\tau}_{\text{net}}^{\text{ext}} = 0, \text{ then } \frac{d\vec{L}}{dt} = 0
\]

\[
\vec{L} = \text{const}
\]
Example: Bullet strikes cylinder edge

Application of Conservation of angular momentum

A bullet of mass $m$ moving with velocity $v$ strikes and becomes embedded at the edge of a cylinder of mass $M$ and radius $R_0$. The cylinder, initially at rest, begins to rotate about its symmetry axis, which remains fixed in position. Assuming no frictional torque, what is the angular velocity of the cylinder after this collision? Is kinetic energy conserved?
Example:

Bullet strikes a cylinder

Given: \( M, \sigma, R \)
Find: \( \omega_f \)

We have a system of two objects and their total angular momentum is conserved:

\[ L_{\text{in}} = L_{\text{fin}} \]

\[ (L_{\text{m}} + L_{\text{M}})_{\text{init}} = (L_{\text{m}} + L_{\text{M}})_{\text{fin}}. \]

\[ L_{\text{m}} = I \cdot \hat{r} \cdot \hat{r} = 0 \]
\[ L_{\text{m}} = \int r \times \vec{p} = rp \sin \theta \cdot \hat{k} = k \cdot r \cdot p \cdot \sin (\theta - \alpha) = k \cdot p \cdot r \cdot \hat{s} \cdot \hat{d} = p \cdot R \cdot \hat{k} = mR \cdot \hat{k} \]

\[ L_{\text{fin}} = I \cdot \vec{\omega}_f = (I_{\text{m}} + I_{\text{M}}) \cdot \vec{\omega}_f = (mR^2 + \frac{1}{2}MR^2) \cdot \vec{\omega}_f \]

So,

\[ (mR^2 + \frac{1}{2}MR^2) \cdot \vec{\omega}_f = mR \cdot \hat{k} \]
\[ \vec{\omega}_f = \frac{mV \cdot \hat{R}}{mR^2 + \frac{1}{2}MR^2} \cdot \hat{k} = \frac{mV \cdot \hat{R}}{(m + \frac{1}{2}M)R} \]
Bicycle wheel/turntable as a demo of Angular Momentum Conservation
Angular Momentum Conservation (z comp): \( L_z^{\text{fin}} = L_z^{\text{in}} = I_z^{\text{fin}} \)

\[
L_z^{\text{fin}} = L_z^{w} + L_z^{AD} = 0 \\
\Rightarrow L_z^{AD} = -L_z^{w} \\
\Rightarrow I_{AD} \omega_{AD} = -I_{w} \omega_{w}
\]

clockwise \( \omega_{AD} = -\left( \frac{I_{w}}{I_{AD}} \right) \omega_{w} \) 

counterclockwise
ConcepTest 1  

Spinning Bicycle Wheel

You are holding a spinning bicycle wheel while standing on a stationary turntable. If you suddenly flip the wheel over so that it is spinning in the opposite direction, the turntable will:

1) remain stationary
2) start to spin in the same direction as before flipping
3) start to spin in the same direction as after flipping

\[ L^\text{in}_z = L^\text{fin}_z \]

\[ L = -L + L_{\text{me+table}} \]

\[ L_{\text{me+table}} = 2L \]

The total angular momentum of the system is \( L \) upward, and it is conserved. So if the wheel has \(-L\) downward, you and the table must have \(+2L\) upward.
Angular Momentum Conservation. Examples

\[ I_1 - \text{large} \quad I_2 - \text{small} \quad \omega_1 - \text{small} \quad \omega_2 - \text{large} \]

For a rigid body
\[ \vec{L} = I \vec{\omega} \]
\[ \vec{L}_1 = \vec{L}_2 \quad I_1 \omega_1 = I_2 \omega_2 \]
\[ \omega_2 = (I_1 / I_2) \omega_1 \]

Flight: leg and hands are in to make \( \omega \) large

Launch: leg and hands are out to make \( I \) large

Landing: leg and hands are out to “dump” large \( \omega \)
Bicycle wheel precession.

A bicycle wheel is spun up to high speed and is suspended from the ceiling by a wire attached to one end of its axle.

Now mg try to tilt the axle downward.

You expect the wheel to go down (and it would if it weren’t rotating), but it unexpectedly swerves to the left and starts rotating!

Let’s explain that.
Bicycle wheel precession.

Let's calculate torque on the wheel
\[ \vec{\tau} = \vec{r} \times m \vec{g} = rmg(\hat{j}) \]
mg produces torque in +y direction

Notice, angular momentum isn’t conserved. There is an external torque produced by mg. So apply Rotational N.2nd law:
\[ \frac{d\vec{L}}{dt} = \vec{\tau} \]
\[ d\vec{L} = \vec{\tau} dt = (rmg dt) \hat{j} \text{ in } +y \text{ direction} \]

\[ \vec{L}_{fin} = \vec{L}_{in} + d\vec{L} = \vec{L}_{in} + \vec{\tau} dt \]
Torque changes direction of angular momentum.

So L rotates.

Now, since \[ \vec{L} = I \vec{\Omega} \]
then angular momentum rotates with angular velocity \( \Omega \)

It is called precession

Top view of the precession
A figure skater spins with her arms extended. When she pulls in her arms, she reduces her rotational inertia and spins faster so that her angular momentum is conserved. Compared to her initial rotational kinetic energy, her rotational kinetic energy after she pulls in her arms must be:

\[ KE_{\text{rot}} = \frac{1}{2} I \omega^2 = \frac{1}{2} (I \omega) \omega = \frac{1}{2} L \omega \]

(used \( L = I \omega \)). Because \( L \) is conserved, larger \( \omega \) means larger \( KE_{\text{rot}} \).

The “extra” energy comes from the work she does on her arms.

**A)** the same

**B)** larger because she’s rotating faster

**C)** smaller because her rotational inertia is smaller
Conservation of Angular Momentum.

Kepler’s second law

Kepler’s second law states that each planet moves so that a line from the Sun to the planet sweeps out equal areas in equal times. Use conservation of angular momentum to show this.
Kepler’s second law

Let’s find ang. moment. of the Earth (treat the Earth as a point):

\[ \vec{L} = \vec{r} \times \vec{p} \]

\[ |\vec{L}| = rp \sin \theta = mrv \sin \theta \]

Next step. Let’s find area swept by \( r \) in \( dt \):

\[ dA = \frac{1}{2} r (v dt) \sin (\pi - \theta) = \frac{1}{2} rv dt \sin \theta \]

\[ \frac{dA}{dt} = \frac{mr v \sin \theta}{2m} = \frac{L}{2m} \]

Let’s show that \( L \) is conserved:

\[ \vec{\tau}_{ext} = \vec{r} \times F_g (-\vec{r}) = 0 \]

\[ \frac{d\vec{L}}{dt} = \vec{\tau}_{net} \]

since \( \vec{\tau}_{net} = 0 \), then \( \vec{L} = \text{const} \)

So if \( L \) is constant, \( dA/dt \) is a constant

Kepler’s 2nd law is a consequence of conservation of ang. momentum!
Thank you
See you on Wednesday