Lecture 6

Chapter 4

Rotational Motion

Course website:
http://faculty.uml.edu/Andriy_Danylov/Teaching/PhysicsI
Today we are going to discuss:

Chapter 4:

- Uniform Circular Motion: Section 4.4
- Nonuniform Circular Motion: Section 4.6
In addition to translation, objects can rotate.

There is rotation everywhere you look in the universe, from the nuclei of atoms to spiral galaxies.

Need to develop a vocabulary for describing rotational motion.
In order to describe rotation, we need to define

How to measure angles?

You know degrees, but in the scientific world **RADIANS** are more popular. Let’s introduce them.
Angular Position in polar coordinates

Consider a pure rotational motion: an object moves around a fixed axis.

Instead of using \( x \) and \( y \) cartesian coordinates, we will define object’s position with: \( r, \theta \)

\[
\theta = \frac{s}{r} \quad \theta \text{ in radians!}
\]

Its definition as a ratio of two length makes it a pure number without dimensions. So, the radian is dimensionless and there is no need to mention it in calculations.

(Thus the unit of angle (radians) is really just a name to remind us that we are dealing with an angle).

If angle is given in radians, we can get an arclength spanning angle \( \theta \)

\[
S = r \theta
\]

Use Radians to get an arc length

\[
\text{CORRECT} \quad S = \frac{\pi}{3} r
\]

\[
\text{INCORRECT} \quad s = 60^\circ r
\]
Examples: angles in radians

\[ s = \text{arc length} = \frac{2\pi r}{4} = \frac{\pi r}{2} \]

Apply

\[ \theta = \frac{s}{r} = \frac{\pi r}{2} \]

\[ \theta = \frac{\pi}{2} \text{ radians!} \]

\[ s = 2\pi r \]

\[ \theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \text{ rad} \]

\[ 360^\circ = 2\pi \text{ rad} \]

\[ 1 \text{ rad} = \frac{360}{2\pi} \approx 57.3^\circ \]
Now we need to introduce rotational kinematic quantities like we did for translational motion.

*Angular displacement, Angular velocity, Angular acceleration* for rotational kinematic equations
Angular displacement and velocity

Angular displacement:
\[ \Delta \theta = \theta_2 - \theta_1 \]

The average angular velocity is defined as the total angular displacement divided by time:
\[ \bar{\omega} = \frac{\Delta \theta}{\Delta t} \]

The instantaneous angular velocity:
\[ \omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt} \]

Angular velocity is the rate at which particle’s angular position is changing.

\[ \text{rad}; \text{deg}; \text{rev}; \text{rev/min} = \text{rpm} \]

For both points, \( \Delta \theta \) and \( \Delta t \) are the same so
\( \omega \) is the same for all points of a rotating object.

That is why we can say that Earth’s angular velocity is \( 7.2 \times 10^{-5} \text{ rad/sec} \) without connecting to any point on the Earth. All points have the same \( \omega \).

So \( \omega \) is like an intrinsic property of a solid rotating object.
Sign of Angular Velocity

When is the Angular velocity positive/negative?

\( \omega \) is positive for a counterclockwise rotation.

\( \omega \) is negative for a clockwise rotation.

As shown in the figure, \( \omega \) can be positive or negative, and this follows from our definition of \( \theta \).

(Definition of \( \theta \): An angle \( \theta \) is measured (convention) from the positive x-axis in a counterclockwise direction.)

Example:

\( \omega \) is negative for a clockwise rotation.

For a clock hand, the angular velocity is negative

\[ \omega < 0 \]
The angular velocity $\omega$ of any point on a solid object rotating about a fixed axis is the same. Both Bonnie and Klyde go around one revolution (2$\pi$ radians) every 2 sec.

ConcepTest

Bonnie sits on the outer rim of a merry-go-round, and Klyde sits midway between the center and the rim. The merry-go-round makes one complete revolution every 2 seconds.

Klyde’s angular velocity is:

- A) same as Bonnie’s
- B) twice Bonnie’s
- C) half of Bonnie’s
- D) one-quarter of Bonnie’s
- E) four times Bonnie’s

$\omega$ is the same for both rabbits

\[
\omega = \frac{\Delta \theta}{\Delta t} = \frac{1\text{ rev}}{2\text{ sec}} = \frac{2\pi \text{ rad}}{2\text{ sec}} = \pi \text{ rad/sec}
\]
Now, if a rotation is not uniform (angular velocity is not constant), we can introduce angular acceleration

\[ \omega \neq \text{const} \]

Angular acceleration

The angular velocity is changing.
Angular Acceleration

The angular acceleration is the rate at which the angular velocity changes with time:

\[ \omega_2 - \omega_1 = \Delta \omega = \frac{\omega_2 - \omega_1}{t_2 - t_1} \]

Average angular acceleration:

\[ \alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} \]

The units of angular acceleration are rad/s²

Since \( \omega \) is the same for all points of a rotating object, angular acceleration also will be the same for all points.

Thus, \( \omega \) and \( \alpha \) are properties of a rotating object.
The Sign of Angular Acceleration, $\alpha$

**Speeding up ccw**
- Initial $\omega > 0$
- Final $\omega > 0$
- $\alpha > 0$
- $\omega_f - \omega_i > 0$
  \[ \alpha = \frac{\omega_f - \omega_i}{t_f - t_i} > 0 \]
- Positive and larger

**Slowing down ccw**
- Initial $\omega > 0$
- Final $\omega < 0$
- $\alpha < 0$
- $\omega_f - \omega_i < 0$
  \[ \alpha = \frac{\omega_f - \omega_i}{t_f - t_i} < 0 \]
- Positive and smaller

**Speeding up cw**
- Initial $\omega < 0$
- Final $\omega > 0$
- $\alpha > 0$
- $\omega_f - \omega_i > 0$
  \[ \alpha = \frac{\omega_f - \omega_i}{t_f - t_i} > 0 \]
- Positive and larger

**Slowing down cw**
- Initial $\omega < 0$
- Final $\omega > 0$
- $\alpha < 0$
- $\omega_f - \omega_i < 0$
  \[ \alpha = \frac{\omega_f - \omega_i}{t_f - t_i} < 0 \]
- Positive and smaller

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**So, $\alpha$ is positive if $|\omega|$ is increasing and is counter-clockwise.**

**a is negative if $|\omega|$ is decreasing and $\omega$ is clockwise.**
The fan blade is slowing down. What are the signs of $\omega$ and $\alpha$?

A) $\omega$ is positive and $\alpha$ is positive.
B) $\omega$ is positive and $\alpha$ is negative.
C) $\omega$ is negative and $\alpha$ is positive.
D) $\omega$ is negative and $\alpha$ is negative.
E) $\omega$ is positive and $\alpha$ is zero.

1) $\omega$ is negative (rotation CW)
2) $\omega$ is slowing down ($|\omega_f| < |\omega_i|$)

For example

\[ \alpha = \frac{-2 - (-5)}{3 - 2} = +3 \text{ rad/s}^2 \]

Case 3 (the previous slide)
Now, since we have introduced all angular quantities, we can write down

**Rotational Kinematic Equations**

For motion with constant angular acceleration

\[ \alpha = \text{const} \]
Rotational kinematic equations

The equations of motion for translational and rotational motion (for constant acceleration) are identical.

Applies to particles with circular trajectories and to rotating solid objects.

**Translational kinematic equations**  
\( \alpha = \text{const} \)

1. \( v = v_o + at \)
2. \( x = x_o + v_o t + \frac{1}{2} at^2 \)
3. \( v^2 = v_o^2 + 2\alpha(x - x_o) \)

**Rotational kinematic equations**  
\( \alpha = \text{const} \)

1. \( v \rightarrow \omega \quad \omega = \omega_o + \alpha t \)
2. \( x \rightarrow \theta \quad \theta = \theta_o + \omega_o t + \frac{1}{2} \alpha t^2 \)
3. \( a \rightarrow \alpha \quad \omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o) \)

\( \omega \) is the slope of \( \theta \)
\( \alpha \) is the slope of \( \omega \)
For a rotating object we can also introduce a linear velocity which is called the **Tangential velocity**

Now we need to introduce a useful expression relating **linear velocity and angular velocity**
Relation between tangential and angular velocities

Each point on a rotating rigid body has \textbf{the same angular} displacement, velocity, and acceleration!

The corresponding \textit{linear (or tangential) variables depend on the radius} and the linear velocity is greater for points farther from the axis.

\[ v_t = r \omega \]

By definition, linear velocity:

In the 1st slide, we defined:

\[ v_t = \frac{ds}{dt} = \left\| \frac{d\theta}{dt} \right\| = r \frac{d\theta}{dt} = \left\| \omega \right\| = r\omega \]

Remember it!!!!
You will use it often!!!

Relation between linear and angular velocities (\textbf{\( \omega \) in rad/sec})
**ConcepTest**  
**Bonnie and Klyde II**

Bonnie sits on the outer rim of a merry-go-round, and Klyde sits midway between the center and the rim. The merry-go-round makes one revolution every 2 seconds. **Who has the larger linear (tangential) velocity?**

A) Klyde  
B) Bonnie  
C) both the same  
D) linear velocity is zero for both of them

We already know that all points of a rotating body have the same angular velocity $\omega$.

**But their linear speeds $v$ will be different because** $v_t = r\omega$ and Bonnie is located farther out (larger radius $R$) than Klyde. $V_{Klyde} = \frac{1}{2}V_{Bonnie}$
For a rotating object we can also introduce the **Acceleration**

**Tangential acceleration**

**Centripetal acceleration**

Now we need to introduce a useful expression relating *linear acceleration and angular acceleration*. 

\[ a = r \alpha \]
Tangential acceleration

The particle in the figure is moving along a circle and is speeding up.

(Definition) **Tangential acceleration** is the rate at which the tangential velocity changes, \( a_t = \frac{dv_t}{dt} \).

\[
a_t = \frac{dv_t}{dt} = ||v_t|| = r \omega = r \frac{d\omega}{dt} = r \alpha
\]

There is a tangential acceleration \( a_t \) which is always tangent to the circle.

There is also the **centripetal acceleration** is \( a_r = \frac{v_t^2}{r} \), where \( v_t \) is the tangential speed.

Finally, any object that is undergoing circular motion experiences two accelerations: centripetal and tangential.

Let’s get a total acceleration:

\[
\vec{a}_{total} = \vec{a}_t + \vec{a}_r \quad a_{total} = \sqrt{a_t^2 + a_r^2}
\]
Centripetal acceleration

- In uniform circular motion ($\omega=\text{const}$), although the speed is constant, there is an acceleration because the direction of the velocity vector is always changing.
- The acceleration of uniform circular motion is called centripetal acceleration.
- The direction of the centripetal acceleration is toward the center of the circle.
- The magnitude of the centripetal acceleration is constant for uniform circular motion:

$$a_r = \frac{v_t^2}{r} \quad (\text{toward center of circle})$$

Centripetal acceleration can be rewritten in term of angular velocity, $\omega$

$$a_r = \frac{v_t^2}{r} = \omega^2 r \quad v_{\tan} = r \omega$$
A car is traveling around a curve at a steady 45 mph. Is the car accelerating?

A) Yes  
B) No  

There is a Centripetal acceleration
A car is traveling around a curve at a steady 45 mph. Which vector shows the direction of the car’s acceleration?

There is a **Centripetal acceleration** pointing toward the center.
A car is slowing down as it drives over a circular hill.

Which of these is the acceleration vector at the highest point?

- **A.** Acceleration (slowing down) of changing speed
- **B.** Acceleration of changing direction
- **C.** Acceleration (slowing down) of changing direction
- **D.** Acceleration of changing speed
- **E.** Acceleration of changing speed
Uniform circular motion

A particle moves with uniform circular motion if its angular velocity is constant.

The time interval to complete one revolution is called the period, T.

The period T is related to the speed v:

\[ v = \frac{1 \text{ circumference}}{1 \text{ period}} = \frac{2\pi r}{T} \]

In this case, as the particle goes around a circle one time, its angular displacement is \( \Delta \theta = 2\pi \) during one period \( \Delta t = T \). Then, the angular velocity is related to the period of the motion:

\[ \omega = \frac{d\theta}{dt} = \frac{\Delta \theta}{\Delta t} = \frac{2\pi}{T} \]

\[ |\omega| = \frac{2\pi \text{ rad}}{T} \quad \text{or} \quad T = \frac{2\pi \text{ rad}}{|\omega|} \]
Thank you
See you on Wednesday

Bye Bye For Now