A geodesic on a surface is a curve, connecting two given points that any nearby curve with the same endpoints is longer. The cylindrical coordinates will be used. Then, the curve is given by
\[
\begin{cases}
  \rho = R \\
  \varphi = \varphi(z) \\
  z = z
\end{cases}
\]
where \( z \) is independent variable.

The length of the infinitesimal segment is given by
\[
ds = \sqrt{\left(\frac{\partial \rho}{\partial z}\right)^2 + \rho^2 \left(\frac{\partial \varphi}{\partial z}\right)^2 + 1} = \sqrt{R^2 \left(\frac{\partial \varphi}{\partial z}\right)^2 + 1}
\]

The length of the curve is
\[
\Lambda = \int \sqrt{1 + R^2 \varphi'^2} \, dz
\]

The curve, that minimizes \( \Lambda \), has to satisfy the Euler-Lagrange equation
\[
\frac{\partial f}{\partial \varphi} - \frac{d}{dz} \frac{\partial f}{\partial \varphi'} = 0
\]

Since \( \frac{\partial f}{\partial \varphi} = 0 \Rightarrow \frac{\partial f}{\partial \varphi'} = C_1 = \text{const} \)

\[
\frac{R^2 \varphi'}{\sqrt{1 + R^2 \varphi'^2}} = C_1 \Rightarrow \varphi'^2 = C_2 (1 + R^2 \varphi'^2) \Rightarrow \varphi'^2 = C_3 \Rightarrow \varphi' = C_4 = \frac{d\varphi}{dz}
\]

so \( \varphi(z) = A \varphi + B \); \( A, B \) - constants of integration, which are determined from the boundary conditions (points 1 and 2). This path spirals around the cylinder from point 1 \((R, \varphi_1, z_1)\) to point 2 \((R, \varphi_2, z_2)\).
The geodesic is not unique. 
First, we can spiral from 1 to 2 clockwise or counterclockwise around the cylinder, making less than one revolution.

Second, we could spiral less steeply, making one or more complete revolutions before arriving to 2.

- If we unwrap and flatten the cylinder, the spiral paths become straight lines, which are known to be the shortest paths on a flat surface.
6.12 Show that the path \( y = y(x) \) for which the integral
\[
\int_{x_i}^{x_f} x \cdot \sqrt{1-y'^2} \, dx
\]
is stationary is an arcsinh function.

The functional is given
\[
S = \int_{x_i}^{x_f} x \cdot \sqrt{1-y'^2} \, dx
\]
The integrand is the function \( f \) for the Eq. of
\[
f[y, y', x] = x \cdot \sqrt{1-y'^2}
\]

\( \text{Eq: } \frac{\partial f}{\partial y} = \frac{df}{dx} \frac{\partial y}{\partial y'} \\
\]
Since there is no \( y \) dependence,
\[
\frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial y'} = \text{constant} = \kappa_1
\]
\[
\frac{\partial f}{\partial y'} = \frac{x \cdot (-2y')}{x \cdot \sqrt{1-y'^2}} = -\frac{x y'}{\sqrt{1-y'^2}} = \kappa_1 \quad \Rightarrow \quad \frac{x y'}{\sqrt{1-y'^2}} = \kappa_1 (\text{constant})
\]
\[
x^2 y'^2 = \kappa_1^2 (1-y'^2)
\]
\[
y'^2 (x^2 + \kappa_1^2) = \kappa_1^2
\]
\[
y' = \frac{dy}{dx} = \frac{k}{\sqrt{x^2 + \kappa_1^2}}
\]
\[
y(x) = \int \frac{k \, dx}{\sqrt{x^2 + \kappa_1^2}} = \int \frac{k \cdot d(\sqrt{x^2 + \kappa_1^2})}{\sqrt{x^2 + \kappa_1^2}}
\]
\[
= \kappa \int \frac{du \, dz}{\sqrt{1 + \sqrt{1 + 8u^2}}}
\]
\[
y(x) = k \cdot \text{arcsinh} \left( \frac{x}{\kappa} \right) + C
\]
Show that the shortest path between two given points in a plane is a straight line, using plane polar coordinates.

We'll use polar coordinates \((r, \theta)\).

An element of length is
\[
\ell = \sqrt{\rho^2 + \rho^2 \frac{d\theta}{d\rho}^2}
\]

Let's treat \(\rho\) as an independent variable: \(\theta = \theta(\rho)\), so
\[
\ell = \sqrt{\rho^2 + \rho^2 \left(\frac{d\theta}{d\rho}\right)^2} \, d\rho
\]

The length is
\[
L = \int \ell \, d\rho = \int \sqrt{1 + \rho^2 \left(\frac{d\theta}{d\rho}\right)^2} \, d\rho
\]

So
\[
f(\rho, \theta, \rho) = \sqrt{1 + \rho^2 \left(\frac{d\theta}{d\rho}\right)^2}
\]

The Euler equation:
\[
\frac{\partial f}{\partial \theta} = \frac{d}{d\rho} \frac{\partial f}{\partial \theta}
\]

The equation is simplified significantly since there's no \(\theta\) dependence.

\[
\frac{\partial f}{\partial \rho} = 0 \Rightarrow \frac{\partial f}{\partial \rho} = \text{const} = K
\]

\[
\frac{\rho^2 \theta'}{2 \sqrt{1 + \rho^2 \theta'^2}} = K \Rightarrow \rho^2 \theta' = K^2 (1 + \rho^2 \theta'^2)
\]

\[
\theta' = \sqrt{\frac{K^2}{\rho^4 - K^2}} = \frac{K}{\rho \sqrt{\rho^2 - K^2}} = \frac{d\theta}{d\rho}
\]

Let's integrate at
\[
\theta = \int \frac{K}{\rho \sqrt{\rho^2 - K^2}} \, d\rho = \frac{K}{\sqrt{\rho^2 - K^2}} = \frac{1}{\rho} \int \frac{du}{\sqrt{4u^2 - 1}} = -\frac{1}{\sqrt{1 - u^2}}
\]

\[
\rho = \cos^{-1} \frac{1}{u} \Rightarrow \rho = \sqrt{\rho_0^2 (1 - \sin^2 \frac{\theta}{2})} = \int \frac{u^2}{\sqrt{1 - u^2}} \, du = \theta + \pi = \cos^{-1} \frac{1}{u} = \sqrt{\rho_0^2 (1 - \sin^2 \theta)}
\]
\[ \psi - \psi_0 = \arccos \left( \frac{k}{\rho} \right) \]

\[ \rho = \frac{k}{\cos(\psi - \psi_0)} \]

- This is an equation of a straight line in polar coordinates.

Let's show that this is true (a straight line) by construction.

1. Put a point \( B \) with coord. \((\rho_0, \psi_0)\)
2. Draw a line (red one) \( \perp \) to \( OB \)
3. Pick an arbitrary point \( P(\rho, \psi) \) on the line.

4. Then, by construction

\[ OB = OP \cdot \cos \angle BOP, \text{ i.e. } OB = \rho_0; \quad OP = \rho, \quad \angle BOP = \psi - \psi_0 \]

\[ \rho_0 = \rho \cdot \cos(\psi - \psi_0) \]

\[ \rho(\psi) = \frac{\rho_0}{\cos(\psi - \psi_0)} \]

And that is our eqn if \( k = \rho_0 \).
The surface is generated by rotating the curve \( y = y(x) \) around the \( x \) axis. The boundary conditions are:

\[
\begin{align*}
  y(x_1) &= y_1, \\
  y(x_2) &= y_2.
\end{align*}
\]

Now, we need to minimize not a line, but a surface. Slicing the surface up into vertical rings, we see that the area is given by

\[
dA = 2\pi y \cdot ds = 2\pi y \cdot \sqrt{dx^2 + dy^2} = 2\pi y \cdot \sqrt{1 + x'^2} dy
\]

The total area is

\[
A = 2\pi \int y \cdot \sqrt{1 + x'^2} dy = \text{functional}
\]

\[
f[x, x', y] = y \cdot \sqrt{1 + x'^2}, \quad x' = dx/dy
\]

\text{Eqn: } \frac{df}{dx} = \frac{d}{dy} \frac{df}{x'} \]

Since \( \frac{df}{dx} = 0 \) \( \Rightarrow \frac{df}{x'} = \text{constant} = y_0 \]

\[
\frac{df}{x'} = Y_0 \cdot \frac{x'}{\sqrt{1 + x'^2}} = y_0 \Rightarrow y_0 x'^2 = y_0^2 (1 + x'^2) \Rightarrow x'^2 (y_0^2 - y_0^2) = y_0^2
\]

\[
x' = \frac{y_0}{\sqrt{y_0^2 - y_0^2}} = \frac{dx}{dy}
\]

\[
x = \int \frac{y_0 \, dy}{\sqrt{y_0^2 - y_0^2}} = y_0 \int \frac{d(y/y_0)}{\sqrt{(y/y_0)^2 - 1}} = \left| \frac{y}{y_0} = \coth \, u \right| \frac{y_0}{d(\frac{y}{y_0})} = \coth \, u \right| = y_0 \int \frac{\sinh \, u \, du}{\sqrt{\coth^2 \, u - 1}} \]

\[
= y_0 \cdot \coth^{-1} \left( \frac{y}{y_0} \right) + x_0; \quad x_0 - \text{the second root of integration.}
\]

\[
x - x_0 = y_0 \cdot \coth^{-1} \left( \frac{y}{y_0} \right) \Rightarrow y = y_0 \cdot \coth \left( \frac{x - x_0}{y_0} \right)
\]