7.1 Write down the Lagrangian for a projectile (subject to no air resistance) in terms of its Cartesian coordinates \((x, y, z)\), with \(z\) measured vertically upward. Find the three Lagrange equations and show that they are exactly what you would expect for the equations of motion.

Assume that \(U = 0\) in the xy plane, so PE is \(U = mgz\).

KE is \(T = \frac{1}{2}m(x^2 + y^2 + z^2)\).

\[ L = T - U = \frac{1}{2}m(x^2 + y^2 + z^2) - mgz \]

The Lagrange eq-ns are

\[
\frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \quad \frac{\partial L}{\partial y} = \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} \quad \frac{\partial L}{\partial z} = \frac{d}{dt} \frac{\partial L}{\partial \dot{z}}
\]

\[ 0 = m\ddot{x} \quad 0 = m\ddot{y} \quad -mg = m\ddot{z} \]

which are the three components of \(F = m\ddot{r}\) for a projectile with \(F = mg\).

7.4 Consider a mass \(m\) moving in a frictionless plane that slopes at an angle \(\alpha\) with the horizontal. Write down the Lagrangian in terms of coordinates \(x\), measured horizontally across the slope, and \(y\), measured down the slope. (Treat the system as two-dimensional, but include the gravitational potential energy.) Find the two Lagrange equations and show that they are what you should have expected.

\[ T = \frac{1}{2}m(x^2 + y^2) - KE \text{ of the block.} \]

Assume \(U = 0\) at the top of the plane, then

\[ U(x, y) = -mgh = -mg \cdot y \cdot \sin \alpha \]

\[ L = T - U = \frac{1}{2}m(x^2 + y^2) + mg \cdot y \cdot \sin \alpha \]

Then, the two Lagrange eq-ns are

\[
\frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \quad \frac{\partial L}{\partial y} = \frac{d}{dt} \frac{\partial L}{\partial \dot{y}}
\]

\[ 0 = m\ddot{x} \quad mg \cdot \sin \alpha = m\ddot{y} \]

The acceleration down the slope is \(g \cdot \sin \alpha\).
Write down $h$ for a 1-D particle moving along the $x$ axis and subject to a force $F = -kx$ (with $k$ positive). Find the Lagrange eqn of motion and solve it.

The KE of the particle is

$$T = \frac{1}{2}m\dot{x}^2$$

The PE is:

$$F = -kx \Rightarrow \Delta T = \int F \cdot dx = W \quad (\text{Eq. 4.7}) \; ; \; U(x) = -W \quad (\text{work})$$

$$U(x) = -\int F \cdot dx, \; \text{so}$$

$$U(x) = -\int (-kx) \cdot dx = \frac{kx^2}{2}$$

The Lagrangian is

$$\mathcal{L} = T - U$$

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{kx^2}{2}$$

The Lagrange eqn is

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \Rightarrow -kx = \frac{d}{dt} \left[ m\dot{x} \right] = m\ddot{x}$$

$$\ddot{x} + \frac{k}{m} x = 0 \quad \text{denote} \; \omega = \sqrt{\frac{k}{m}}$$

The solution is

$$x(t) = A \cos(\omega t - \delta)$$

where $A, \delta$ - arbitrary constants.
Lagrange’s equations in the form discussed in this chapter hold only if the forces (at least the nonconstraint forces) are derivable from a potential energy. To get an idea how they can be modified to include forces like friction, consider the following: A single particle in one dimension is subject to various conservative forces (net conservative force \( F = -\frac{\partial U}{\partial x} \)) and a nonconservative force (let’s call it \( F_{\text{nc}} \)). Define the Lagrangian as \( \mathcal{L} = T - U \) and show that the appropriate modification is

\[
\frac{\partial \mathcal{L}}{\partial x} + F_{\text{nc}} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}.
\]

The Lagrangian of a single particle in 1D space is

\[
\mathcal{L} = T - U = \frac{1}{2} m \dot{x}^2 - U(x)
\]

Let’s look at these derivatives:

\[
\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F \quad \text{(conservative force)}
\]

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = m \ddot{x}
\]

Now, we can use these in N. 2nd law

\[
\sum F = F + F_{\text{nc}} = m \ddot{x}, \quad \text{so}
\]

\[
\frac{\partial \mathcal{L}}{\partial x} + F_{\text{nc}} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}
\]
7.23 A small cart (mass $m$) is mounted on rails inside a large cart. The two are attached by a spring (force constant $k$) in such a way that the small cart is in equilibrium at the midpoint of the large. The distance of the small cart from its equilibrium is denoted $x$ and that of the large one from a fixed point on the ground is $X$, as shown in Figure 7.13. The large cart is now forced to oscillate such that $X = A \cos \omega t$, with both $A$ and $\omega$ fixed. Set up the Lagrangian for the motion of the small cart and show that the Lagrange equation has the form

$$\ddot{x} + \omega_n^2 x = B \cos \omega t$$

where $\omega_n$ is the natural frequency $\omega_n = \sqrt{k/m}$ and $B$ is a constant. This is the form assumed in Section 5.5, Equation (5.57), for driven oscillations (except that we are here ignoring damping). Thus the system described here would be one way to realize the motion discussed here.

- $x_1$, position of a small cart relative to the inertial RF
  $$x_1 = X + x$$

  Be careful, we are asked to write $x$ for the small cart, not the whole system of the two carts. $X = A \cos \omega t$, so the big cart is driven with something.

  So, the small cart velocity:
  $$v = \dot{x}_1 = \dot{X} + \dot{x} = \omega A \sin \omega t - A \omega \cos \omega t - \dot{A} \sin \omega t$$

  The KE of the small cart:
  $$T = \frac{1}{2} m \dot{x}_1^2 = \frac{1}{2} m (\omega A \sin \omega t - A \omega \cos \omega t)^2 = \frac{m \omega^2 A^2}{2} \sin^2 \omega t - m \omega A^2 \sin \omega t \cos \omega t$$

  The PE is
  $$U = \frac{1}{2} k x^2$$

  $$L = T - U = \frac{m \dot{x}_1^2}{2} + \frac{m \omega^2 A^2}{2} \sin^2 \omega t - m \omega A \sin \omega t \cos \omega t - \frac{1}{2} k x^2$$

- $\frac{dL}{dx} = \frac{d}{dt} \frac{d}{dx}$
  $$-kx = \frac{d}{dt} [m \dot{x} - m A \omega \sin \omega t] = m \ddot{x} - m A \omega^2 \cos \omega t \quad \bigg| \bigg|_{m} \quad \ddot{x} + \frac{k}{m} x = A \omega^2 \cos \omega t$$

- driven oscillations.

- $\ddot{x} + \omega_n^2 x = B \cos \omega t$, denote $\frac{1}{m} = \omega_n^2$; $B = A \omega^2$

  the natural frequency.
7.31 A simple pendulum (mass $M$ and length $L$) is suspended from a cart (mass $m$) that can oscillate on the end of a spring of force constant $k$, as shown in Figure 7.15. (a) Write the Lagrangian in terms of the two generalized coordinates $x$ and $\phi$, where $x$ is the extension of the spring from its equilibrium length. (Read the hint in Problem 7.29.) Find the two Lagrange equations. (Warning: They're pretty ugly!) (b) Simplify the equations to the case that both $x$ and $\phi$ are small. (They're still pretty ugly, and note, in particular, that they are still coupled; that is, each equation involves both variables. Nonetheless, we shall see how to solve these equations in Chapter 11—see particularly Problem 11.19.)

The system with 2 degrees of freedom $\Rightarrow x, \phi$—generalized coord.

The Cartesian coord. of $M$ is $(X, Y)$

\[
\begin{align*}
\dot{X} &= x + L \cdot \sin \phi \\
\dot{Y} &= -L \cdot \cos \phi
\end{align*}
\]

Assume $\dot{Y} = 0$ at $Y = 0$.

The PE of the bob: $U_m = -mgL \cdot \cos \phi$.

The PE of the cart: $U_m = \frac{1}{2} kx^2$.

The KE of the system:

\[
T = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} M \left( \dot{x}^2 + \dot{y}^2 \right)
\]

\[
= \frac{m}{2} \dot{x}^2 + \frac{M}{2} \left[ (\dot{x} + L \dot{\phi} \cos \phi)^2 + L^2 \dot{\phi}^2 \sin^2 \phi \right]
\]

\[
= \frac{m}{2} \dot{x}^2 + \frac{M}{2} \left[ \dot{x}^2 + L^2 \dot{\phi}^2 \cos^2 \phi + 2L \dot{x} \dot{\phi} \cos \phi + L^2 \dot{\phi}^2 \sin^2 \phi \right]
\]

\[
= \frac{1}{2} (m + M) \dot{x}^2 + \frac{M}{2} \left[ L^2 \dot{\phi}^2 + 2L \dot{x} \dot{\phi} \cos \phi \right]
\]

\[
L = T - U = \frac{1}{2} (m + M) \dot{x}^2 + \frac{M}{2} \left( L^2 \dot{\phi}^2 + 2L \dot{x} \dot{\phi} \cos \phi \right) + mgL \cdot \cos \phi - \frac{kx^2}{2}
\]

Now, the two Lagrange eqn's

\[
\frac{\partial h}{\partial x} = \frac{d}{dt} \frac{\partial h}{\partial \dot{x}}
\]

\[
-kx = \frac{d}{dt} \left[ (m + M) \dot{x} + mL \dot{\phi} \cos \phi \right] = (m + M) \ddot{x} + mL \ddot{\phi} \cos \phi + mL \dot{\phi}^2 \sin \phi
\]

(1) \[ -kx = (m + M) \ddot{x} + mL \ddot{\phi} \cos \phi - mL \dot{\phi}^2 \sin \phi \]
\[ \frac{d^2 \theta}{dt^2} = d \frac{d \theta}{dt} \phi \]

\[-ML \dot{\phi} \sin \theta - Mg L \sin \theta = \frac{d}{dt} \left[ M L^2 \ddot{\phi} + ML \dot{x} \cos \theta \right] = M L^2 \dddot{\phi} + ML \ddot{x} \cos \theta - ML \dot{x} \dot{\phi} \sin \theta \]

\[
\begin{align*}
\begin{cases}
L \dddot{\theta} + \dddot{x} \cos \theta & = -g \sin \theta \\
(m+M) \dddot{x} + ML \dddot{\phi} \cos \theta - ML \dddot{\phi} \sin \theta & = -k \chi
\end{cases}
\end{align*}
\]

\{ \text{coupled ODE.} \}

5) If \( \theta \) remains small, we can simplify the eqns

\[
\begin{align*}
\cos \theta & \approx 1 \\
\sin \theta & \approx \theta
\end{align*}
\]

\[
\begin{align*}
(1) \quad (m+M) \dddot{x} + ML \dddot{\phi} - ML \dddot{\phi} \approx -k \chi \\
(2) \quad L \dddot{\theta} + \dddot{x} & = -g \phi
\end{align*}
\]
7.35 ** Figure 7.16 is a bird’s-eye view of a smooth horizontal wire loop that is forced to rotate at a fixed angular velocity $\omega$ about a vertical axis through the point A. A bead of mass $m$ is threaded on the loop and is free to move around it, with its position specified by the angle $\phi$ that it makes at the center with the diameter AB. Find the Lagrangian for this system using $\phi$ as your generalized coordinate. (Read the hint in Problem 7.29.) Use the Lagrange equation of motion to show that the bead oscillates about the point B exactly like a simple pendulum. What is the frequency of these oscillations if their amplitude is small?

\[
\begin{align*}
(x_1, y_1) & \quad \text{defining position of wire} \\
(x_2, y_2) & \quad \text{position of point 0.} \\
\dot{x_2} & = R \cdot \cos \omega t \\
\dot{y_2} & = R \cdot \sin \omega t \\
\dot{x_1} & = R \cdot \cos \omega t + R \cdot \cos (\omega t + \phi) \\
\dot{y_1} & = y + R \cdot \sin (\omega t + \phi) \\
\ddot{x_2} & = -R \cdot \omega \cdot \sin \omega t \\
\ddot{y_2} & = R \cdot \omega \cdot \cos \omega t \\
\ddot{x_1} & = -R \cdot \omega \cdot \sin \omega t - R \cdot \sin (\omega t + \phi) \cdot \sin (\omega t + \phi) \\
\ddot{y_1} & = R \cdot \omega \cdot \cos \omega t + R \cdot (\omega + \phi) \cdot \sin (\omega t + \phi) \\
T & = \frac{1}{2} m (\dot{x_2}^2 + \dot{y_2}^2) = \frac{mR^2}{2} \left[ (\omega \cdot \sin \omega t - (\omega + \phi) \sin (\omega t + \phi))^2 + (\omega \cdot \cos \omega t + (\omega + \phi) \cdot \cos (\omega t + \phi))^2 \right] = \\
& = \frac{mR^2}{2} \left[ \omega^2 \sin^2 \omega t + (\omega + \phi)^2 \sin^2 (\omega t + \phi) + 2 \omega (\omega + \phi) \cdot \sin (\omega t + \phi) \cdot \sin \omega t + \\
& + \omega^2 \cos^2 \omega t + (\omega + \phi)^2 \cos^2 (\omega t + \phi) + 2 \omega (\omega + \phi) \cdot \cos (\omega t + \phi) \cdot \cos \omega t \right] = \\
& = \frac{mR^2}{2} \left[ \omega^2 + (\omega + \phi)^2 + 2 \omega (\omega + \phi) \cdot (\omega + \phi) \cdot \cos \omega t + \sin (\omega t + \phi) \cdot \sin \omega t \right] \\
& \| \cos (2 \cdot \phi) = \cos \phi \cdot \cos \phi + \sin \phi \cdot \sin \phi \| \\
& \text{is used}
\end{align*}
\]

If you are comfortable with your speculations, you can skip these calculations and get $T$ in a different way.
The angular velocity of O is \( \omega \), so the velocity of O relative to A is \( \mathbf{v}_O = \omega \mathbf{R} \).

Then, the angular velocity of M is \( \omega + \dot{\phi} \), so the velocity of M relative to O is \( \mathbf{v}_{MO} = R(\omega + \dot{\phi}) \).

Thus, the velocity of M relative to A is
\[
\mathbf{v}_M = \mathbf{v}_O + \mathbf{v}_{MO}
\]

The KE is
\[
T = \frac{m}{2} (\mathbf{v}_M \cdot \dot{\mathbf{v}}_M) = \frac{m}{2} (\mathbf{v}_O + \mathbf{v}_{MO})^2 + 2 \mathbf{v}_O \cdot \mathbf{v}_{MO}
\]

An angle between \( \mathbf{v}_O \) and \( \mathbf{v}_{MO} \) is \( \psi \) (see the figure), so
\[
T = \frac{mR^2}{2} \left( \omega^2 R^2 + (\omega + \dot{\phi})^2 + 2 \omega R (\omega + \dot{\phi}) \cos \psi \right)
\]

T = \(- \frac{mR^2}{2} \left( \omega^2 (\omega + \dot{\phi})^2 + 2 \omega (\omega + \dot{\phi}) \cos \psi \right)
\]

so we get exactly the same expression.

But!!! Be careful, it's very easy to make mistakes using the second method. The first one is tedious but much more reliable.

Because the hoop is horizontal, the PE is constant, and we may as well take it to be \( U = 0 \), so
\[
L = T - U = T = \frac{mR^2}{2} \left[ \omega^2 + (\omega + \dot{\phi})^2 + 2 \omega (\omega + \dot{\phi}) \cos \psi \right]
\]

\[
\frac{\partial L}{\partial \dot{\psi}} = \frac{d}{dt} \frac{\partial L}{\partial \psi}
\]

\[\frac{\partial}{\partial \psi} (\frac{mR^2}{2} (\omega + \dot{\phi}) \cdot \sin \psi) = \frac{d}{dt} \left[ \frac{mR^2}{2} (\omega + \dot{\phi}) + 2 \omega \cos \psi \right]
\]

\[
\dot{\psi} + \omega^2 \sin \psi = 0
\]

which is the eqn of a simple pendulum oscillating about the point \( \psi = 0 \) (point B). The frequency of \( \omega \) matches \( \omega = \omega \).