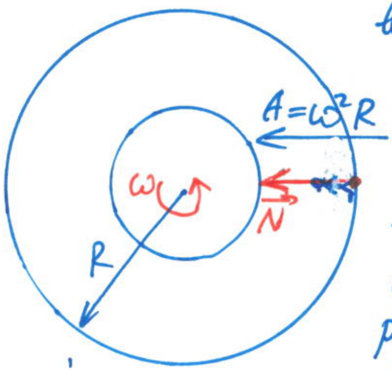
9.2

9.2★ A donut-shaped space station (outer radius R) arranges for artificial gravity by spinning on the axis of the donut with angular velocity ω. Sketch the forces on, and accelerations of, an astronaut standing in the station **(a)** as seen from an inertial frame outside the station and **(b)** as seen in the astronaut's personal rest frame (which has a centripetal acceleration $A = \omega^2 R$ as seen in the inertial frame). What angular velocity is needed if $R = 40$ meters and the apparent gravity is to equal the usual value of about 10 m/s^2? **(c)** What is the percentage difference between the perceived g at a six-foot astronaut's feet ($R = 40$ m) and at his head ($R = 38$ m)?

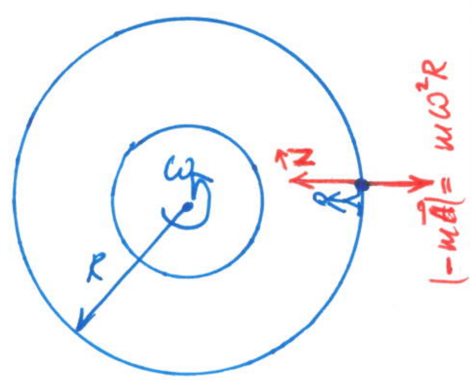
(a)



INERTIAL RF

A centripetal force makes the astronaut follow the circular path and it is equal to the normal force $\vec{N}$. ($\vec{N}=\vec{F}_{cp}$). So, the guy experiences a centripetal acceleration ($\omega^2 R$) provided by the normal force.
(seen by an inertial observer)

(b)



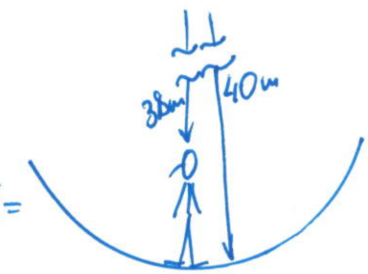
Astronaut's frame

As seen by an observer inside the station, the austronaut is at rest under the action of two forces, the normal force $\vec{N}$ and the force $-m\vec{A}$

To simulate gravity, we need

$$A=\omega^2 R = g \Rightarrow \omega = \sqrt{g/R} = \approx \sqrt{\frac{10\,m/s^2}{40\,m}} = 0.5\,rad/s = 4.8\,rpm.$$

(c)



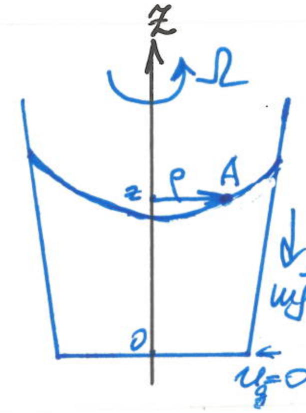
$$\delta(\%) = \frac{g^{feet}_{fake} - g^{head}_{fake}}{g^{feet}_{fake}} \cdot 100\% =$$

$$= \frac{\omega^2 (R_{feet} - R_{Head})}{\omega^2 \cdot R_{feet}} \cdot 100\% = \frac{40-38}{40} \cdot 100\% = 5\%$$

9.14

9.14 ** I am spinning a bucket of water about its vertical axis with angular velocity Ω. Show that, once the water has settled in equilibrium (relative to the bucket), its surface will be a parabola. (Use cylindrical polar coordinates and remember that the surface is an equipotential under the combined effects of the gravitational and centrifugal forces.)

Consider the problem in the rotating frame of the bucket.



The centrifugal force pushes water to "climb" up the wall of the bucket, but the gravity drags water down. As a result we'll have a certain shape of water. We expect that the free-surface of the water is not flat; it's depressed in the middle and rises up to its highest point along the rim of the bucket.

The easiest way to solve it is an energy approach.

Consider an arbitrary point A on the water surface. The water is in equilibrium and its surface is an equipotential surface for the combined gravitational and centrifugal force. Let's find PEs associated with these forces.

$$U = -\int_1^2 \vec{F}\cdot d\vec{r}$$

$$U_{cf} = -\int \vec{F}_{cf}\cdot d\vec{\rho}' = -\int_0^{\rho} (m\Omega^2\rho)\hat{\rho}\cdot(d\rho'\cdot\hat{\rho}) = -m\Omega^2\int_0^{\rho}\rho' d\rho' = -m\Omega^2\frac{\rho^2}{2}$$

$$U_g = -\int_1^2 \vec{F}_g\cdot d\vec{z}' = -\int_0^z (mg)\cdot(-\hat{z})\cdot dz\cdot\hat{z} = mgz$$

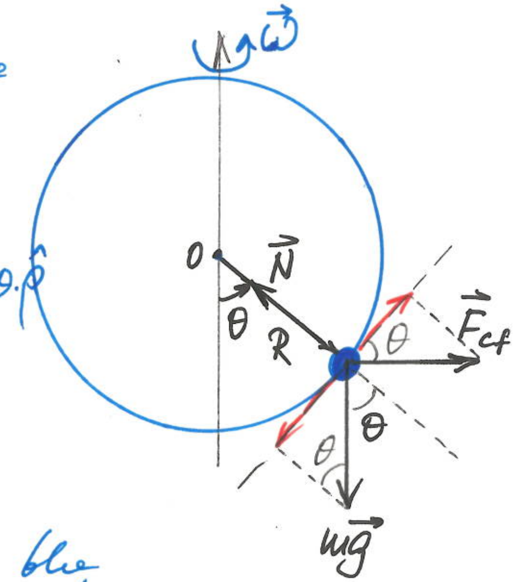
So, the surface is defined by

$$mgz - \frac{m\Omega^2\rho^2}{2} = const$$

$$\boxed{z = \frac{\Omega^2\rho^2}{2g} + const}$$ - which is a parabola

9.17

9.17 ★ Consider the bead threaded on a circular hoop of Example 7.6 (page 260), working in a frame that rotates with the hoop. Find the equation of motion of the bead, and check that your result agrees with Equation (7.69). Using a free-body diagram, explain the result (7.71) for the equilibrium positions.



Consider the problem in a refer. frame of a rotating hoop. There are three forces acting on the bead:

Normal force − $\vec N$

Centrifugal force − $\vec F_{cf} = m(\vec\omega \times \vec r)\times\vec\omega = m\omega^2 R\cdot\sin\theta\cdot\hat\rho$

Gravity − $\vec F_g = m\vec g$

Sorry! One more!

Coriolis force − $\vec F_c$

- But since the Coriolis force acts $\perp$ to the plane of the hoop, it will be balanced by a $\perp$ component of the normal force. So, we can forget about them.
- Moreover, the radial components of the forces cancel each other also.

$$\vec N\cdot\hat r = \vec F_{cf}\cdot\hat r + m\vec g\cdot\hat r$$

So, they need not concern us.

- The bead can move only in the tangential direction. Its eq-n is

$$ma_t = \sum_i F_t \text{, so}$$

$$mR\ddot\theta = F_{cf}\cdot\cos\theta - mg\cdot\sin\theta = (m\omega^2 R\cdot\sin\theta)\cos\theta - mg\cdot\sin\theta$$

$$\boxed{\ddot\theta = (\omega^2\cdot\cos\theta - g/R)\cdot\sin\theta}$$ which is exactly Eq-n. 7.69.

- When the beat in equilibrium, $(\vec F_{cf})_{tan} = (m\vec g)_{tan}$, i.e $\ddot\theta = 0$, so

$$F_{cf}\cdot\cos\theta = mg\cdot\sin\theta$$

$$\cancel{m}\omega^2 R\cdot\cancel{\sin\theta}\cdot\cos\theta = \cancel{m}g\cdot\cancel{\sin\theta} \Rightarrow \cos\theta = g/\omega^2 R$$

$$\boxed{\theta = \pm\arccos\left(g/\omega^2 R\right)}$$ This is the same as 7.71.

9.26

9.26 ** In Section 9.8, we used a method of successive approximations to find the orbit of an object that is dropped from rest, correct to first order in the earth's angular velocity Ω. Show in the same way that if an object is thrown with initial velocity $\mathbf{v}_o$ from a point O on the earth's surface at colatitude θ, then to first order in Ω its orbit is

$$\left.\begin{aligned} x &= v_{xo}t + \Omega(v_{yo}\cos\theta - v_{zo}\sin\theta)t^2 + \tfrac{1}{3}\Omega g t^3 \sin\theta \\ y &= v_{yo}t - \Omega(v_{xo}\cos\theta)t^2 \\ z &= v_{zo}t - \tfrac{1}{2}gt^2 + \Omega(v_{xo}\sin\theta)t^2. \end{aligned}\right\} \qquad (9.73)$$

[First solve the equations of motion (9.53) in zeroth order, that is, ignoring Ω entirely. Substitute your zeroth-order solution for $\dot{x}$, $\dot{y}$, and $\dot{z}$ into the right side of equations (9.53) and integrate to give the next approximation. Assume that v_o is small enough that air resistance is negligible and that $\mathbf{g}$ is a constant throughout the flight.]

The eq-n of motion on the earth surface

$$m\ddot{\vec{r}} = m\vec{g}_0 + \vec{F}_{cf} + \vec{F}_{cor} \ ; \quad \vec{F}_{eff} = m\vec{g}_0 + \vec{F}_{cf} = m\vec{g}$$

$$\ddot{\vec{r}} = \vec{g} + \frac{\vec{F}_{cor}}{m} = \vec{g} + \frac{2m(\dot{\vec{r}} \times \vec{\Omega})}{m} = \vec{g} + 2\dot{\vec{r}} \times \vec{\Omega}$$

since it depends on $\dot{\vec{r}}, \ddot{\vec{r}}$ we can move the coord. system from the earth center to the surface

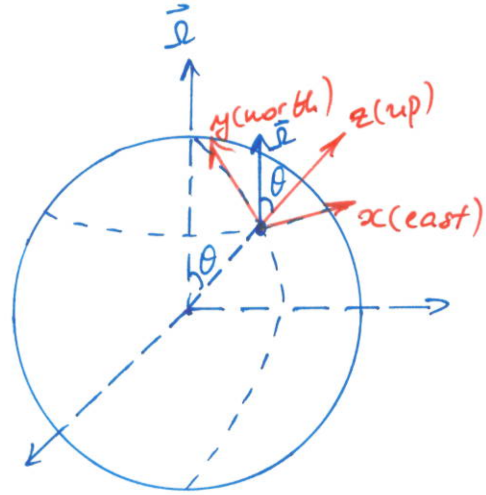
$\hat{x}$ (east), $\hat{y}$ (north), $\hat{z}$ (up)

In this coord. system

$\vec{V}_0 = (v_{0x}, v_{0y}, v_{0z})$

$\vec{\Omega} = (0, \Omega \cdot \sin\theta, \Omega \cdot \cos\theta)$

$\vec{g} = (0, 0, -g)$



$$\ddot{\vec{r}} = \vec{g} + 2\dot{\vec{r}} \times \vec{\Omega}$$

solve it using perturbation analysis.

power series in a small parameter Ω:

$$\vec{r}(t) = \vec{r}_0(t) + \Omega\vec{r}_1(t) + \Omega^2\vec{r}_2(t) + \dots$$

put it in the DE:

$$\underline{\ddot{\vec{r}}_0} + \underline{\underline{\Omega\ddot{\vec{r}}_1}} + \underline{\underline{\underline{\Omega^2\ddot{\vec{r}}_2}}} + \dots = \underline{\vec{g}} + 2\left(\underline{\dot{\vec{r}}_0} + \underline{\underline{\Omega\dot{\vec{r}}_1}} + \Omega^2\dot{\vec{r}}_2 + \dots\right) \times \vec{\Omega}$$

The corresponding terms must be equal, so

(1) $\dfrac{d^2\vec{r}_0}{dt^2} = \vec{g}$ (2) $\dfrac{d^2\vec{r}_1}{dt^2} = 2\dfrac{d\vec{r}_0}{dt}\times\hat{\Omega}$, $\hat{\Omega} \equiv \dfrac{\vec{\Omega}}{\Omega}$

solve (1)

$\Rightarrow \dot{\vec{r}}_0 = \vec{g}t + \vec{v}_0 \Rightarrow \vec{r}_0(t) = \vec{g}\dfrac{t^2}{2} + \vec{v}_0 t + \vec{r}_0(0)$

put it into (2),

$$\frac{d^2\vec{r}_1}{dt^2} = 2(\vec{g}t + \vec{v}_0)\times\vec{\Omega}$$

$$\dot{\vec{r}}_1 = (\vec{g}\times\hat{\Omega})t^2 + 2(\vec{v}_0\times\hat{\Omega})t$$

$\vec{r}_1(t) = (\vec{g}\times\hat{\Omega})\dfrac{t^3}{3} + (\vec{v}_0\times\hat{\Omega})t^2$, so the final solution is

$$\boxed{\vec{r}(t) = \vec{r}_0 + \Omega\vec{r}_1 = \frac{\vec{g}t^2}{2} + \vec{v}_0 t + \vec{r}(0) + (\vec{g}\times\vec{\Omega})\frac{t^3}{3} + (\vec{v}_0\times\vec{\Omega})t^2}$$

the eq-n that can be used for ANY initial conditions, let's apply to ours

$\vec{r}(0) = 0$; $\vec{v}_0 = (v_{0x}, v_{0y}, v_{0z})$

$$(\vec{g}\times\vec{\Omega}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & -g \\ 0 & \Omega\cdot\sin\theta & \Omega\cdot\cos\theta \end{vmatrix} = (g\Omega\cdot\sin\theta, 0, 0)$$

$$(\vec{v}_0\times\vec{\Omega}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_{0x} & v_{0y} & v_{0z} \\ 0 & \Omega\cdot\sin\theta & \Omega\cos\theta \end{vmatrix} = \hat{x}(v_{0y}\Omega\cos\theta - v_{0z}\Omega\cdot\sin\theta) - \hat{y}(v_{0x}\Omega\cos\theta) + \hat{z}\,v_{0x}\Omega\cdot\sin\theta.$$

so now we can decompose $\vec{r}(t)$ in $\hat{x}, \hat{y}, \hat{z}$

$$\vec{r}(t) = -\frac{gt^2}{2}\hat{z} + v_{0x}t\,\hat{x} + v_{0y}t\cdot\hat{y} + v_{0z}t\cdot\hat{z} + g\Omega\cdot\sin\theta\cdot\hat{x}\cdot\frac{t^3}{3} +$$
$$+ \hat{x}t^2(v_{0y}\Omega\cos\theta - v_{0z}\Omega\cdot\sin\theta) - \hat{y}t^2(v_{0x}\Omega\cdot\cos\theta) + \hat{z}t^2 v_{0x}\Omega\cdot\sin\theta$$

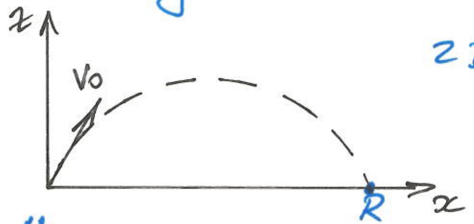
$$\begin{cases} x(t) = v_{0x}\cdot t + \Omega(v_{0y}\cdot\cos\theta - v_{0z}\sin\theta)\cdot t^2 + \frac{1}{3}\Omega g t^3\cdot\sin\theta \\ y(t) = v_{0y}t - \Omega(v_{0x}\cdot\cos\theta)t^2 \\ z(t) = v_{0z}t - \dfrac{gt^2}{2} + \Omega\cdot(v_{0x}\cdot\sin\theta)t^2 \end{cases}$$

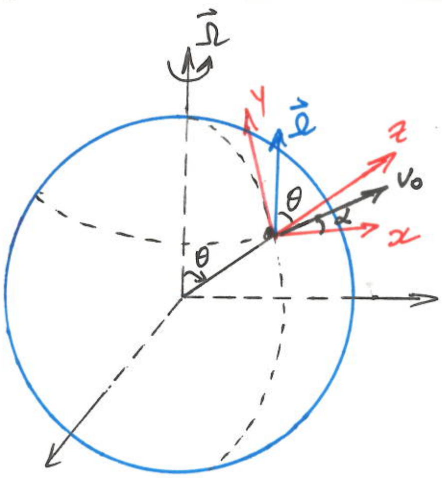
9.28

9.28 ** Use the result (9.73) of Problem 9.26 to do the following: A naval gun shoots a shell at colatitude θ in a direction that is α above the horizontal and due east, with muzzle speed v_o. **(a)** Ignoring the earth's rotation (and air resistance), find how long (t) the shell would be in the air and how far away (R) it would land. If $v_o = 500$ m/s and $\alpha = 20°$, what are t and R? **(b)** A naval gunner spots an enemy ship due east at the range R of part (a) and, forgetting about the Coriolis effect, aims his gun exactly as in part (a). Find by how far north or south, and in which direction, the shell will miss the target, in terms of Ω, v_o, α, θ, and g. (It will also miss in the east–west direction but this is perhaps less critical.) If the incident occurs at latitude 50° north ($\theta = 40°$), what is this distance? What if the latitude is 50° south? (This problem is a serious issue in long-range gunnery: In a battle near the Falkland Islands in World War I, the British navy consistently missed German ships by many tens of yards because they apparently forgot that the Coriolis effect in the southern hemisphere is opposite to that in the north.)

(a) $\vec{\Omega} = 0$ (ignore the earth's rotation)

2D problem.





$$m\ddot{\vec{r}} = \sum \vec{F} = mg(-\hat{z})$$

(x) $\ddot{x} = 0$

$x = At + B$ apply the i.c.

$x(t) = V_{0x}\cdot t + x_0$

$x(t) = V_0\cdot\cos\alpha\cdot t = V_{0x} t$

(z) $m\ddot{z} = -mg$

$z(t) = -\frac{gt^2}{2} + Ct + D$

$z(t) = -\frac{gt^2}{2} + V_{0z}\cdot t + z_0$

$z(t) = -\frac{gt^2}{2} + V_0\cdot\sin\alpha\cdot t$

$z = 0 \Rightarrow 0 = t\left(-\frac{gt}{2} + V_0\sin\alpha\right)$

$t_1 = 0$; $\boxed{t_2 = \frac{2V_0\cdot\sin\alpha}{g} = \frac{2\cdot V_{0z}}{g}}$ ← that's how long the shell would be in the air.

Range? $x(t_2) = V_0\cdot\cos\alpha\cdot\left(\frac{2V_0\cdot\sin\alpha}{g}\right) = \boxed{\frac{2V_0^2\cdot\sin\alpha\cdot\cos\alpha}{g} = \frac{2V_{0x}\cdot V_{0z}}{g} = R}$

if $V_0 = 500$ m/s; $\alpha = 20°$, then

$t_2 = 34.9$ s; $R = 16.4$ km.

(b) Here we need to apply eq-ns (9.73) from Problem 9.26 to our initial conditions:

$$\vec{V}_0 = (V_0 \cdot \cos\alpha, 0, V_0 \cdot \sin\alpha)$$
$$\vec{\Omega} = (0, \Omega \cdot \sin\theta, \Omega \cdot \cos\theta)$$
$$\vec{g} = (0, 0, -g)$$

$$\begin{cases} x(t) = V_0 \cos\alpha\, t + \Omega(0 \cdot \cos\theta - V_0 \cdot \sin\alpha \cdot \sin\theta)t^2 + \frac{1}{3}\Omega \cdot g t^3 \cdot \sin\theta \\ y(t) = 0 \cdot t - \Omega \cdot V_0 \cdot \cos\alpha \cdot \cos\theta \cdot t^2 \\ z(t) = V_0 \cdot \sin\alpha\, t - \frac{gt^2}{2} + \Omega \cdot V_0 \cdot \cos\alpha \cdot \sin\theta \cdot t^2 \end{cases}$$

$$x(t) = \left[V_0 \cdot \cos\alpha - \Omega \cdot V_0 \cdot \sin\alpha \cdot \sin\theta\, t + \frac{1}{3}\Omega g t^2 \cdot \sin\theta\right] \cdot t$$
$$\boxed{y(t) = -\Omega \cdot V_0 \cdot \cos\alpha \cdot \cos\theta \cdot t^2}$$
$$z(t) = \left[V_0 \cdot \sin\alpha - \frac{gt}{2} + \Omega \cdot V_0 \cdot \cos\alpha \cdot \sin\theta \cdot t\right] \cdot t$$

y direction determines north/south deflection

If/ $\theta = 40°$ $\Rightarrow y(t) = -(7.3 \cdot 10^{-5}\,1/s) \cdot (500\,m/s) \cdot \cos 20° \cdot \cos 40 \cdot t^2$

$$\begin{cases} \Omega = 7.3 \cdot 10^{-5}\,1/s \\ V_0 = 500\,m/s \\ \alpha = 20° \end{cases}$$

Oops! we still need to find t. We can get it from $z(t)$ eq-n, when $z(t_3) = 0$, so

$$t_3 = \frac{V_0 \cdot \sin\alpha}{\frac{g}{2} - \Omega V_0 \cdot \cos\alpha \cdot \sin\theta} = \frac{V_0 \cdot \sin\alpha}{4.9 - 0.022} \approx 35\,s$$

→ very small correction in comparison with part a.

it's ~~slightly~~ more than 34.9 s from part a.

So, now we can go back to our calculations of $y(t_3)$

$y(t_3) = -32.18\,m$; minus means deflection to the south

- If the latitude is 50° south? It means $\theta = 90° + 50° = 140°$
$\cos 40° = -\cos 140°$, so we'll get extra minus, which means that the shell will be deflected to the north in the southern hemisphere by 32.18 m.