IN THIS CHAPTER, you will learn about systems that oscillate in simple harmonic motion.
What are oscillations?

Oscillatory motion is a repetitive motion back and forth around an equilibrium position. We’ll describe oscillations in terms of their amplitude, period, and frequency. The most important oscillation is simple harmonic motion (SHM), where the position and velocity graphs are sinusoidal.
What things undergo SHM?
The prototype of SHM is a mass oscillating on a spring. Lessons learned from this system apply to all SHM.
- A pendulum is a classic example of SHM.
- Any system with a linear restoring force undergoes SHM.

LOOKING BACK  Section 9.4  Restoring forces
How is SHM related to circular motion?

The projection of uniform circular motion onto a line oscillates back and forth in SHM.

- This link to circular motion will help us develop the mathematics of SHM.
- A phase constant, based on the angle on a circle, will describe the initial conditions.

<< LOOKING BACK Section 4.4 Circular motion
Is energy conserved in SHM?
If there is no friction or other dissipative force, the mechanical energy of an oscillating system is conserved. Energy is transformed back and forth between kinetic and potential energy. Energy conservation is an important problem-solving strategy.

LOOKING BACK  Sections 10.3–10.5  Elastic potential energy and energy diagrams
What if there’s friction?
If there’s dissipation, the system “runs down.” This is called a damped oscillation. The oscillation amplitude undergoes exponential decay. But the amplitude can grow very large when an oscillatory system is driven at its natural frequency. This is called resonance.
Why is SHM important?

Simple harmonic motion is one of the most common and important motions in science and engineering.

- Oscillations and vibrations occur in mechanical, electrical, chemical, and atomic systems. Understanding how a system might oscillate is an important part of engineering design.
- More complex oscillations can be understood in terms of SHM.
- Oscillations are the sources of waves, which we’ll study in the next two chapters.
Oscillatory Motion

- Objects that undergo a repetitive motion back and forth around an equilibrium position are called oscillators.
- The time to complete one full cycle, or one oscillation, is called the period \( T \).
- The number of cycles per second is called the frequency \( f \), measured in Hz:

\[
f = \frac{1}{T} \quad \text{or} \quad T = \frac{1}{f}
\]

\[1 \ \text{Hz} = 1 \ \text{cycle per second} = 1 \ \text{s}^{-1}\]
A particular kind of oscillatory motion is **simple harmonic motion**.

In the figure an air-track glider is attached to a spring.

The glider’s position measured 20 times every second.

The object’s maximum displacement from equilibrium is called the amplitude $A$ of the motion.
Simple Harmonic Motion

- The top image shows position versus time for an object undergoing simple harmonic motion.
- The bottom image shows the velocity versus time graph for the same object.
- The velocity is zero at the times when $x = \pm A$; these are the turning points of the motion.
- The maximum speed $v_{\text{max}}$ is reached at the times when $x = 0$.
If the object is released from rest at time $t = 0$, we can model the motion with the cosine function:

$$x(t) = A \cos(\omega t)$$

Cosine is a *sinusoidal* function.

$\omega$ is called the angular frequency, defined as

$$\omega = \frac{2\pi}{T}$$

The units of $\omega$ are rad/s:

$$\omega = 2\pi f$$
Simple Harmonic Motion

- The position of the oscillator is 

\[ x(t) = A \cos(\omega t) \]

- Using the derivative of the position function, we find the velocity:

\[ v_x(t) = \frac{dx}{dt} = -\frac{2\pi A}{T} \sin\left(\frac{2\pi t}{T}\right) = -2\pi fA \sin(2\pi ft) = -\omega A \sin \omega t \]

- The maximum speed is 

\[ v_{\text{max}} = \omega A \]
Example 15.1 A System in Simple Harmonic Motion

**EXAMPLE 15.1** A system in simple harmonic motion

An air-track glider is attached to a spring, pulled 20.0 cm to the right, and released at $t = 0$ s. It makes 15 oscillations in 10.0 s.

a. What is the period of oscillation?
b. What is the object’s maximum speed?
c. What are the position and velocity at $t = 0.800$ s?

**MODEL** An object oscillating on a spring is in SHM.
Example 15.1 A System in Simple Harmonic Motion

**EXAMPLE 15.1** A system in simple harmonic motion

**SOLVE**

a. The oscillation frequency is

\[ f = \frac{15 \text{ oscillations}}{10.0 \text{ s}} = 1.50 \text{ oscillations/s} = 1.50 \text{ Hz} \]

Thus the period is \( T = 1/f = 0.667 \text{ s} \).

b. The oscillation amplitude is \( A = 0.200 \text{ m} \). Thus

\[ v_{\text{max}} = \frac{2\pi A}{T} = \frac{2\pi (0.200 \text{ m})}{0.667 \text{ s}} = 1.88 \text{ m/s} \]
Example 15.1 A System in Simple Harmonic Motion

**EXAMPLE 15.1** A system in simple harmonic motion

**SOLVE** c. The object starts at $x = +A$ at $t = 0$ s. This is exactly the oscillation described by Equations 15.2 and 15.6. The position at $t = 0.800$ s is

$$x = A \cos \left( \frac{2\pi t}{T} \right) = (0.200 \text{ m}) \cos \left( \frac{2\pi (0.800 \text{ s})}{0.667 \text{ s}} \right)$$

$$= (0.200 \text{ m}) \cos (7.54 \text{ rad}) = 0.0625 \text{ m} = 6.25 \text{ cm}$$

The velocity at this instant of time is

$$v_x = -v_{\max} \sin \left( \frac{2\pi t}{T} \right) = -(1.88 \text{ m/s}) \sin \left( \frac{2\pi (0.800 \text{ s})}{0.667 \text{ s}} \right)$$

$$= -(1.88 \text{ m/s}) \sin (7.54 \text{ rad}) = -1.79 \text{ m/s} = -179 \text{ cm/s}$$

At $t = 0.800$ s, which is slightly more than one period, the object is 6.25 cm to the right of equilibrium and moving to the left at 179 cm/s. Notice the use of radians in the calculations.
Example 15.2 Finding the Time

A mass oscillating in simple harmonic motion starts at \( x = A \) and has period \( T \). At what time, as a fraction of \( T \), does the object first pass through \( x = \frac{1}{2} A \)?

**Solve** Figure 15.3 showed that the object passes through the equilibrium position \( x = 0 \) at \( t = \frac{1}{3} T \). This is one-quarter of the total distance in one-quarter of a period. You might expect it to take \( \frac{1}{8} T \) to reach \( \frac{1}{2} A \), but this is not the case because the SHM graph is not linear between \( x = A \) and \( x = 0 \). We need to use \( x(t) = A \cos(2\pi t/T) \). First, we write the equation with \( x = \frac{1}{2} A \):

\[
\frac{A}{2} = A \cos \left( \frac{2\pi t}{T} \right)
\]

Then we solve for the time at which this position is reached:

\[
t = \frac{T}{2\pi} \cos^{-1} \left( \frac{1}{2} \right) = \frac{T}{2\pi} \frac{\pi}{3} = \frac{1}{6} T
\]

**Assess** The motion is slow at the beginning and then speeds up, so it takes longer to move from \( x = A \) to \( x = \frac{1}{2} A \) than it does to move from \( x = \frac{1}{2} A \) to \( x = 0 \). Notice that the answer is independent of the amplitude \( A \).
Figure (a) shows a “shadow movie” of a ball made by projecting a light past the ball and onto a screen.

As the ball moves in uniform circular motion, the shadow moves with simple harmonic motion.

The block on a spring in figure (b) moves with the same motion.
What if an object in SHM is not initially at rest at \( x = A \) when \( t = 0 \)?

Then we may still use the cosine function, but with a **phase constant** measured in radians.

In this case, the two primary kinematic equations of SHM are:

\[
x(t) = A \cos(\omega t + \phi_0)
\]

\[
\nu_x(t) = -\omega A \sin(\omega t + \phi_0) = -v_{\text{max}} \sin(\omega t + \phi_0)
\]
The Phase Constant

- Oscillations described by different values of the phase constant.

\[ \phi_0 = \frac{\pi}{3} \text{ rad} \]

\[ \phi_0 = -\frac{\pi}{3} \text{ rad} \]

\[ \phi_0 = \pi \text{ rad} \]
**Example 15.3 Using the Initial Conditions**

An object on a spring oscillates with a period of 0.80 s and an amplitude of 10 cm. At $t = 0$ s, it is 5.0 cm to the left of equilibrium and moving to the left. What are its position and direction of motion at $t = 2.0$ s?

**Model** An object oscillating on a spring is in simple harmonic motion.
EXAMPLE 15.3 Using the initial conditions

**SOLVE** We can find the phase constant \( \phi_0 \) from the initial condition \( x_0 = -5.0 \text{ cm} = A \cos \phi_0 \). This condition gives

\[
\phi_0 = \cos^{-1} \left( \frac{x_0}{A} \right) = \cos^{-1} \left( -\frac{1}{2} \right) = \pm \frac{2}{3} \pi \text{ rad} = \pm 120^\circ
\]

Because the oscillator is moving to the *left* at \( t = 0 \), it is in the upper half of the circular-motion diagram and must have a phase constant between 0 and \( \pi \) rad. Thus \( \phi_0 \) is \( \frac{2}{3} \pi \) rad. The angular frequency is

\[
\omega = \frac{2\pi}{T} = \frac{2\pi}{0.80 \text{ s}} = 7.85 \text{ rad/s}
\]
**EXAMPLE 15.3** Using the initial conditions

**SOLVE** Thus the object’s position at time $t = 2.0$ s is

$$x(t) = A \cos(\omega t + \phi_0)$$

$$= (10 \text{ cm}) \cos((7.85 \text{ rad/s})(2.0 \text{ s}) + \frac{2}{3} \pi)$$

$$= (10 \text{ cm}) \cos(17.8 \text{ rad}) = 5.0 \text{ cm}$$

The object is now 5.0 cm to the right of equilibrium. But which way is it moving? There are two ways to find out. The direct way is to calculate the velocity at $t = 2.0$ s:

$$v_x = -\omega A \sin(\omega t + \phi_0) = +68 \text{ cm/s}$$
Example 15.3 Using the Initial Conditions

**EXAMPLE 15.3** Using the initial conditions

**SOLVE** The velocity is positive, so the motion is to the right. Alternatively, we could note that the phase at \( t = 2.0 \) s is \( \phi = 17.8 \) rad. Dividing by \( \pi \) you can see that

\[
\phi = 17.8 \text{ rad} = 5.67\pi \text{ rad} = (4\pi + 1.67\pi) \text{ rad}
\]

The \( 4\pi \) rad represents two complete revolutions. The “extra” phase of \( 1.67\pi \) rad falls between \( \pi \) and \( 2\pi \) rad, so the particle in the circular-motion diagram is in the lower half of the circle and moving to the right.
Energy in Simple Harmonic Motion

- An object of mass $m$ on a frictionless horizontal surface is attached to one end of a spring of spring constant $k$.
- The other end of the spring is attached to a fixed wall.
- As the object oscillates, the energy is transformed between kinetic energy and potential energy, but the mechanical energy $E = K + U$ doesn’t change.
Energy is conserved in Simple Harmonic Motion:

\[ E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \]

\[ E(at \ x = \pm A) = U = -kA^2 \]

\[ E(at \ x = 0) = K = \frac{1}{2}m(v_{\text{max}})^2 \]
Frequency of Simple Harmonic Motion

- In SHM, when $K$ is maximum, $U = 0$, and when $U$ is maximum, $K = 0$.

- $K + U$ is constant, so $K_{\text{max}} = U_{\text{max}}$:
  \[ \frac{1}{2}m(v_{\text{max}})^2 = \frac{1}{2}kA^2 \]

- So $v_{\text{max}} = \sqrt{\frac{k}{m}}A$

- Earlier, using kinematics, we found that
  \[ v_{\text{max}} = \frac{2\pi A}{T} = 2\pi fA = \omega A \]

- So
  \[ \omega = \sqrt{\frac{k}{m}} \quad f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad T = 2\pi \sqrt{\frac{m}{k}} \]
EXAMPLE 15.4  Using conservation of energy

A 500 g block on a spring is pulled a distance of 20 cm and released. The subsequent oscillations are measured to have a period of 0.80 s.

a. At what position or positions is the block’s speed 1.0 m/s?
b. What is the spring constant?

MODEL  The motion is SHM. Energy is conserved.
**Example 15.4 Using Conservation of Energy**

**Solve** a. The block starts from the point of maximum displacement, where $E = U = \frac{1}{2}kA^2$. At a later time, when the position is $x$ and the speed is $v$, energy conservation requires

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

Solving for $x$, we find

$$x = \sqrt{A^2 - \frac{mv^2}{k}} = \sqrt{A^2 - \left(\frac{v}{\omega}\right)^2}$$

where we used $k/m = \omega^2$ from Equation 15.24. The angular frequency is easily found from the period: $\omega = 2\pi/T = 7.85 \text{ rad/s}$. Thus

$$x = \sqrt{(0.20 \text{ m})^2 - \left(\frac{1.0 \text{ m/s}}{7.85 \text{ rad/s}}\right)^2} = \pm 0.15 \text{ m} = \pm 15 \text{ cm}$$

There are two positions because the block has this speed on either side of equilibrium.
**EXAMPLE 15.4** Using conservation of energy

**SOLVE** b. Although part a did not require that we know the spring constant, it is straightforward to find from Equation 15.24:

\[ T = 2\pi \sqrt{\frac{m}{k}} \]

\[ k = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2 (0.50 \text{ kg})}{(0.80 \text{ s})^2} = 31 \text{ N/m} \]
The top set of dots is a motion diagram for SHM going to the right.

The bottom set of dots is a motion diagram for SHM going to the left.

At $x = 0$, the object’s speed is as large as possible, but it is *not changing*; hence acceleration is zero at $x = 0$. 
Acceleration in Simple Harmonic Motion

- Acceleration is the time-derivative of the velocity:

\[ a_x = \frac{dv_x}{dt} = \frac{d}{dt}(-\omega A \sin \omega t) = -\omega^2 A \cos \omega t \]

\[ a_x = -\omega^2 x \]

- In SHM, the acceleration is proportional to the negative of the displacement.
Consider a mass $m$ oscillating on a horizontal spring with no friction. The spring force is

$$ (F_{\text{sp}})_x = -k \Delta x $$

Since the spring force is the net force, Newton’s second law gives

$$ (F_{\text{net}})_x = (F_{\text{sp}})_x = -kx = ma_x $$

$$ a_x = -\frac{k}{m} x $$

Since $a_x = -\omega^2 x$, the angular frequency must $\omega = \sqrt{\frac{k}{m}}$. 
Motion for a mass hanging from a spring is the same as for horizontal SHM, but the equilibrium position is affected.

\[ \Delta L = \frac{mg}{k} \]

The block hanging at rest has stretched the spring by \( \Delta L \).
EXAMPLE 15.6  Bungee oscillations

An 83 kg student hangs from a bungee cord with spring constant 270 N/m. The student is pulled down to a point where the cord is 5.0 m longer than its unstretched length, then released. Where is the student, and what is his velocity 2.0 s later?

MODEL  A bungee cord can be modeled as a spring. Vertical oscillations on the bungee cord are SHM.
**Example 15.6 Bungee Oscillations**

**Example 15.6** Bungee oscillations

**Solve** Although the cord is stretched by 5.0 m when the student is released, this is not the amplitude of the oscillation. Oscillations occur around the equilibrium position, so we have to begin by finding the equilibrium point where the student hangs motionless. The cord stretch at equilibrium is given by Equation 15.40:

$$\Delta L = \frac{mg}{k} = 3.0 \text{ m}$$

Stretching the cord 5.0 m pulls the student 2.0 m below the equilibrium point, so $A = 2.0 \text{ m}$. That is, the student oscillates with amplitude $A = 2.0 \text{ m}$ about a point 3.0 m beneath the bungee cord’s original end point.

The bungee cord is modeled as a spring.
**EXAMPLE 15.6 Bungee Oscillations**

**SOLVE.** The student’s position as a function of time, measured from the equilibrium position, is

\[ y(t) = (2.0 \text{ m}) \cos(\omega t + \phi_0) \]

where \( \omega = \sqrt{k/m} = 1.80 \text{ rad/s} \).

The initial condition

\[ y_0 = A \cos \phi_0 = -A \]

requires the phase constant to be \( \phi_0 = \pi \text{ rad} \). At \( t = 2.0 \text{ s} \) the student’s position and velocity are

\[ y = (2.0 \text{ m}) \cos ((1.80 \text{ rad/s})(2.0 \text{ s}) + \pi \text{ rad}) = 1.8 \text{ m} \]

\[ v_y = -\omega A \sin(\omega t + \phi_0) = -1.6 \text{ m/s} \]

The student is 1.8 m above the equilibrium position, or 1.2 m below the original end of the cord. Because his velocity is negative, he’s passed through the highest point and is heading down.

The bungee cord is modeled as a spring.
The Simple Pendulum

- Consider a mass $m$ attached to a string of length $L$ which is free to swing back and forth.
- If it is displaced from its lowest position by an angle $\theta$, Newton’s second law for the tangential component of gravity, parallel to the motion, is

$$ (F_{\text{net}})_t = \sum F_t = (F_G)_t = -mg \sin \theta = ma_t $$

$$ \frac{d^2 s}{dt^2} = -g \sin \theta $$
If we restrict the pendulum’s oscillations to small angles (< 10°), then we may use the small angle approximation \( \sin \theta \approx \theta \), where \( \theta \) is measured in radians.

\[
(F_{\text{net}})_t = -mg \sin \theta \approx -mg\theta = -\frac{mg}{L}s
\]

and the angular frequency of the motion is found to be

\[
\omega = 2\pi f = \sqrt{\frac{g}{L}}
\]
EXAMPLE 15.7  The maximum angle of a pendulum

A 300 g mass on a 30-cm-long string oscillates as a pendulum. It has a speed of 0.25 m/s as it passes through the lowest point. What maximum angle does the pendulum reach?

MODEL  Assume that the angle remains small, in which case the motion is simple harmonic motion.
Example 15.7 The Maximum Angle of a Pendulum

**Example 15.7** The maximum angle of a pendulum

**Solve** The angular frequency of the pendulum is

\[ \omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8 \text{ m/s}^2}{0.30 \text{ m}}} = 5.72 \text{ rad/s} \]

The speed at the lowest point is \( v_{\text{max}} = \omega A \), so the amplitude is

\[ A = s_{\text{max}} = \frac{v_{\text{max}}}{\omega} = \frac{0.25 \text{ m/s}}{5.72 \text{ rad/s}} = 0.0437 \text{ m} \]

The maximum angle, at the maximum arc length \( s_{\text{max}} \), is

\[ \theta_{\text{max}} = \frac{s_{\text{max}}}{L} = \frac{0.0437 \text{ m}}{0.30 \text{ m}} = 0.146 \text{ rad} = 8.3^\circ \]

**Assess** Because the maximum angle is less than \( 10^\circ \), our analysis based on the small-angle approximation is reasonable.
**Simple harmonic motion**

For any system with a restoring force that’s linear or can be well approximated as linear.

- Motion is SHM around the equilibrium position.
- Frequency and period are independent of the amplitude.
- Mathematically:
  - For an appropriate position variable \( u \), the equation of motion can be written
    \[
    \frac{d^2u}{dt^2} = -Cu
    \]
    where \( C \) is a collection of constants.
  - The angular frequency is \( \omega = \sqrt{C} \).
  - The position and velocity are
    \[
    u = A \cos(\omega t + \phi_0) \quad v_u = -v_{\text{max}} \sin(\omega t + \phi_0)
    \]
    where \( A \) and \( \phi_0 \) are determined by the initial conditions.
  - Mechanical energy is conserved.
- Limitations: Model fails if the restoring force deviates significantly from linear.
The Physical Pendulum

- Any solid object that swings back and forth under the influence of gravity can be modeled as a physical pendulum.
- The gravitational torque for small angles ($\theta < 10^\circ$) is
  \[ \tau = -Mgl\theta \]
- Plugging this into Newton’s second law for rotational motion, $\tau = I\alpha$, we find the equation for SHM, with
  \[ \omega = 2\pi f = \sqrt{\frac{Mgl}{I}} \]
EXAMPLE 15.9 A swinging leg as a pendulum

A student in a biomechanics lab measures the length of his leg, from hip to heel, to be 0.90 m. What is the frequency of the pendulum motion of the student’s leg? What is the period?

MODEL We can model a human leg reasonably well as a rod of uniform cross section, pivoted at one end (the hip) to form a physical pendulum. For small-angle oscillations it will undergo SHM. The center of mass of a uniform leg is at the midpoint, so $l = L/2$. 
EXAMPLE 15.9 A swinging leg as a pendulum

SOLVE The moment of inertia of a rod pivoted about one end is 
\( I = \frac{1}{3}ML^2 \), so the pendulum frequency is 

\[
f = \frac{1}{2\pi} \sqrt{\frac{Mg I}{I}} = \frac{1}{2\pi} \sqrt{\frac{Mg(L/2)}{ML^2/3}} = \frac{1}{2\pi} \sqrt{\frac{3g}{2L}} = 0.64 \text{ Hz}
\]

The corresponding period is \( T = 1/f = 1.6 \text{ s} \). Notice that we didn’t need to know the mass.
Example 15.9 A Swinging Leg as a Pendulum

**EXAMPLE 15.9** A swinging leg as a pendulum

**ASSESS** As you walk, your legs do swing as physical pendulums as you bring them forward. The frequency is fixed by the length of your legs and their distribution of mass; it doesn’t depend on amplitude. Consequently, you don’t increase your walking speed by taking more rapid steps—changing the frequency is difficult. You simply take longer strides, changing the amplitude but not the frequency.
Damped Oscillations

- An oscillation that runs down and stops is called a damped oscillation.
- The shock absorbers in cars and trucks are heavily damped springs.
- The vehicle’s vertical motion, after hitting a rock or a pothole, is a damped oscillation.
- One possible reason for dissipation of energy is the drag force due to air resistance.
- The forces involved in dissipation are complex, but a simple linear drag model is

\[ \vec{F}_{\text{drag}} = -b\vec{v} \]  
(model of the drag force)
Damped Oscillations

- When a mass on a spring experiences the force of the spring as given by Hooke’s Law, as well as a linear drag force of magnitude $|F_{\text{drag}}| = bv$, the solution is

$$x(t) = Ae^{-bt/2m} \cos(\omega t + \phi_0) \quad \text{(damped oscillator)}$$

where the angular frequency is given by

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$$

- Here $\omega_0 = \sqrt{k/m}$ is the angular frequency of the undamped oscillator ($b = 0$).
Damped Oscillations

- Position-versus-time graph for a damped oscillator.

\[ A \text{ is the initial amplitude.} \]

The envelope of the amplitude decays exponentially:

\[ x_{\text{max}} = Ae^{-bt/2m} \]
Damped Oscillations

- A damped oscillator has position $x = x_{\text{max}} \cos(\omega t + \phi_0)$, where
  $$x_{\text{max}}(t) = Ae^{-bt/2m}$$

- This slowly changing function $x_{\text{max}}$ provides a border to the rapid oscillations, and is called the **envelope**.

- The figure shows several oscillation envelopes, corresponding to different values of the damping constant $b$. 
  ![Graph showing oscillation envelopes for different values of $b$](image-url)
Mathematical Aside: Exponential Decay

- **Exponential decay** occurs in a vast number of physical systems of importance in science and engineering.
- Mechanical vibrations, electric circuits, and nuclear radioactivity all exhibit exponential decay.

The graph shows the function:

$$u = Ae^{-v/v_0} = A \exp(-v/v_0)$$

where

- $e = 2.71828\ldots$ is Euler’s number.
- $\exp$ is the exponential function.
- $v_0$ is called the decay constant.
Energy in Damped Systems

- Because of the drag force, the mechanical energy of a damped system is no longer conserved.
- At any particular time we can compute the mechanical energy from
  \[ E(t) = E_0 e^{-t/\tau} \]

- Where the decay constant of this function is called the time constant \( \tau \), defined as
  \[ \tau = \frac{m}{b} \]
- The oscillator’s mechanical energy decays exponentially with time constant \( \tau \).
Driven Oscillations and Resonance

- Consider an oscillating system that, when left to itself, oscillates at a natural frequency $f_0$.

- Suppose that this system is subjected to a periodic external force of driving frequency $f_{\text{ext}}$.

- The amplitude of oscillations is generally not very high if $f_{\text{ext}}$ differs much from $f_0$.

- As $f_{\text{ext}}$ gets closer and closer to $f_0$, the amplitude of the oscillation rises dramatically.

- A singer or musical instrument can shatter a crystal goblet by matching the goblet’s natural oscillation frequency.
The response curve shows the amplitude of a driven oscillator at frequencies near its natural frequency of 2.0 Hz.
The figure shows the same oscillator with three different values of the damping constant.

The resonance amplitude becomes higher and narrower as the damping constant decreases.

A lightly damped system has a very tall and very narrow response curve.

A heavily damped system has little response.
Chapter 15 Summary Slides
Dynamics

SHM occurs when a linear restoring force acts to return a system to an equilibrium position.

**Horizontal spring**

\[(F_{\text{net}})_x = -kx\]

**Vertical spring**

The origin is at the equilibrium position \(\Delta L = \frac{mg}{k}\).

\[(F_{\text{net}})_y = -ky\]

**Both:**

\[\omega = \sqrt{\frac{k}{m}} \quad T = 2\pi \sqrt{\frac{m}{k}}\]

**Simple pendulum**

\[\omega = \sqrt{\frac{g}{L}} \quad T = 2\pi \sqrt{\frac{L}{g}}\]

**Physical pendulum**

\[\omega = \sqrt{\frac{Mgl}{I}} \quad T = 2\pi \sqrt{\frac{I}{Mgl}}\]
Energy

If there is no friction or dissipation, kinetic and potential energy are alternately transformed into each other, but the total mechanical energy \( E = K + U \) is conserved.

\[
E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2
= \frac{1}{2}m(v_{\text{max}})^2
= \frac{1}{2}kA^2
\]

The energy of a lightly damped oscillator decays exponentially

\[
E = E_0 e^{-t/\tau}
\]

where \( \tau \) is the time constant.
Simple harmonic motion (SHM) is a sinusoidal oscillation with period $T$ and amplitude $A$.

**Frequency** $f = \frac{1}{T}$

**Angular frequency**

$$\omega = 2\pi f = \frac{2\pi}{T}$$

**Position** $x(t) = A \cos(\omega t + \phi_0)$

$$= A \cos\left(\frac{2\pi t}{T} + \phi_0\right)$$

**Velocity** $v_x(t) = -v_{\text{max}} \sin(\omega t + \phi_0)$ with maximum speed $v_{\text{max}} = \omega A$

**Acceleration** $a_x(t) = -\omega^2 x(t) = -\omega^2 A \cos(\omega t + \phi_0)$
Important Concepts

SHM is the projection onto the x-axis of uniform circular motion. 
φ = ωt + φ₀ is the phase

The position at time t is

\[ x(t) = A \cos \phi \]
\[ = A \cos(\omega t + \phi_0) \]

The phase constant \( \phi_0 \) is determined by the initial conditions:

\[ x_0 = A \cos \phi_0 \]
\[ v_{0x} = -\omega A \sin \phi_0 \]
Resonance

When a system is driven by a periodic external force, it responds with a large-amplitude oscillation if $f_{\text{ext}} \approx f_0$, where $f_0$ is the system’s natural oscillation frequency, or resonant frequency.
Damping

If there is a drag force $\vec{F}_{\text{drag}} = -b\vec{v}$, where $b$ is the damping constant, then (for lightly damped systems)

$$x(t) = Ae^{-bt/2m} \cos(\omega t + \phi_0)$$

The time constant for energy loss is $\tau = m/b$. 