92.128 Calculus 1A Fall 2011 Final Exam Name SOLUTION Section # ________

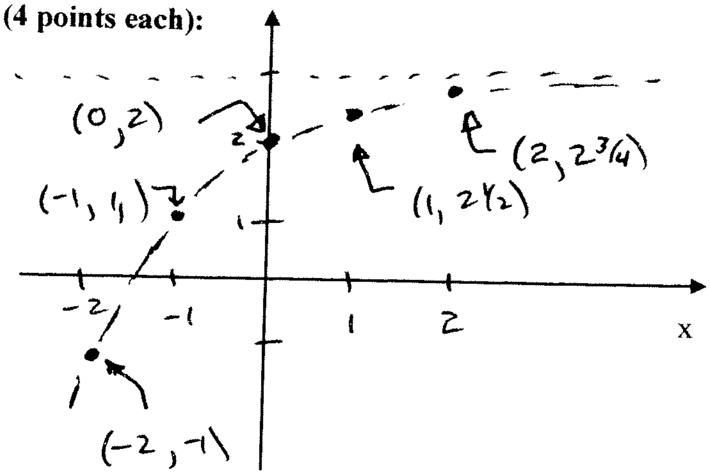
Instructions: This is a TWO hour exam. Note sheets and calculators are **not allowed**. All answers must be supported by work on your exam sheets. Answers with little or no supporting work will receive little or no credit. Graders have been instructed to deduct points if your work is not **neat**, **organized**, **easily interpreted** and **expressed in complete mathematical sentences** (e.g., $f'(x) = something$). Your name and section number must be printed on ALL sheets.

Memorization and pre-calculus questions (4 points each):

M.1 Plot $f(x) = 3 - 2^{-x}$
You MUST plot and label at least four points.

List your four points here:

x	$f(x)$
-2	-1
-1	1
0	2
1	$2\frac{1}{2}$
2	$2\frac{3}{4}$



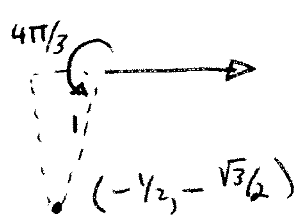
M.2 You discover a lake that is the shape of a perfect circle with radius equal to $\frac{3}{4}$ mile. If you walk π miles around its perimeter, how many radians measured from the lake's center did your walk cover?

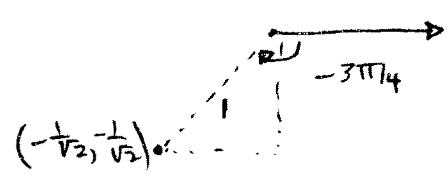
$S = r\theta \qquad \pi = \frac{3}{4}\theta \Rightarrow \theta = 4\pi/3$

M.3 Provide the exact values of:

$\sec(4\pi/3) = -2$

$\tan(-3\pi/4) = \dfrac{-1/\sqrt{2}}{-1/\sqrt{2}} = 1$





M.4 What is the ratio of the area of a circle having a radius of $r_{circle} = 2$ to the surface area of a sphere having a radius of $r_{sphere} = 3$?

$A_{circle} = \pi r^2 = 4\pi \qquad A_{sphere} = 4\pi r^2 = 36\pi \qquad \dfrac{A_{circle}}{A_{sphere}} = \dfrac{4\pi}{36\pi} = \dfrac{1}{9}$

1. a. (4 points) Determine the **domain** of the function $g(x) = \dfrac{\sqrt{x^3+27}}{x^2-5x-6}$. Express your final answer in interval notation.

$x^3+27 \geq 0$

$x^3 \geq -27$

$x \geq -3$

$x^2-5x-6 = (x-6)(x+1) \neq 0$

$x \neq -1$ or 6

PLACE FINAL ANSWER HERE

Domain: $x \in [-3,-1) \cup (-1,6) \cup (6,\infty)$

b. (4 points) Determine the **range** of the function $h(x) = \dfrac{5^x - \pi}{3} + 2$. Express your answer in interval notation.

$5^x > 0$

$\Rightarrow h(x) > -\frac{\pi}{3}+2$

PLACE FINAL ANSWER HERE

Range: $h(x) \in (2-\frac{\pi}{3}, \infty)$

2. (6 points) Determine the amplitude, period and frequency of $r(t) = -2\cos(2t/3-4)+\pi$. Also, what is the maximum value of $r(t)$? Each answer will be scored "all or nothing."

$A = |-2| = 2$

$\frac{2t_p}{3} = 2\pi \Rightarrow t_p = 3\pi$

$f = \frac{1}{t_p} = \frac{1}{3\pi}$

$r_{max} = \pi + 2$

PLACE FINAL ANSWERS HERE

Amplitude = 2

Period = 3π

Frequency = $1/3\pi$

$r_{MAX}(t) =$ $\pi+2$

3. a. (6 points) Simplify the expression $A = \left(\dfrac{a^{3/5} b^{-1/6} c^{-2}}{a^{-1/2} b^{5/3} c^{-8}}\right)^{-1/3}$ using only positive exponents. **Note:** Each exponent will be scored "all or nothing" (2 points each).

$$A = \left[a^{3/5-(-\frac{1}{2})}\, b^{-1/6-5/3}\, c^{-2-(-8)}\right]^{-1/3}$$

$$= \left[a^{11/10}\, b^{-11/6}\, c^{6}\right]^{-1/3}$$

$$= a^{-11/30}\, b^{+11/18}\, c^{-2}$$

PLACE FINAL ANSWER HERE

$A = \dfrac{b^{11/18}}{a^{11/30}\, c^{2}}$

4. (8 Points; 2 point each) Answer the following questions about $f(x) = \dfrac{4x^2 - 4x - 3}{3 - x^2}$

a. What is the value of the y-intercept?
b. What is the value of the x-intercept(s)?
c. Find the vertical asymptote(s).
d. Does f(x) have a horizontal asymptote? If so, what is its value?

a. $y_{\text{intercept}} = f(0) = \dfrac{-3}{3} = -1$

b. $4x^2 - 4x - 3 = 0$

$$x = \frac{4 \pm \sqrt{16+48}}{8}$$

$$= \frac{1}{2} \pm 1 = -\frac{1}{2},\ 3/2$$

c. $3 - x^2 = 0 \quad x = \pm\sqrt{3}$

d. Yes, $y = \dfrac{4}{-1} = -4$

PLACE FINAL ANSWERS HERE

a. y-intercept: $y = -1$

b. x-intercept(s): $x = -1/2$ and $3/2$

c. Vertical asymptote(s): $X = \pm\sqrt{3}$

d. Horizontal asymptote (if any): $y = -4$

5. (14 points) Use the definition of the derivative (**the limit of the difference quotient**) to determine the derivative of $g(x) = \dfrac{1}{x^2+1}$. Note: You should check your answer using the rules of differentiation, but no points are given for rule-based differentiation.

$$\frac{dg}{dx} = \lim_{h \to 0} \frac{\dfrac{1}{(x+h)^2+1} - \dfrac{1}{x^2+1}}{h}$$

$$= \lim_{h \to 0} \frac{(x^2+1) - [(x^2+2xh+h^2)+1]}{h[(x+h)^2+1][x^2+1]}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + \cancel{1} - \cancel{x^2} - 2x\cancel{h} - h^{\cancel{2}} - \cancel{1}}{\cancel{h}[(x+h)^2+1][x^2+1]}$$

$$= \lim_{h \to 0} \frac{-2x-h}{[(x+h)^2+1][x^2+1]}$$

$$= \frac{-2x}{(x^2+1)^2} = -2x(x^2+1)^{-2}$$

checks

$$g(x) = (x^2+1)^{-1} \Rightarrow \frac{dg}{dx} = -(x^2+1)^{-2} \cdot 2x$$

6. Differentiate the following (5 points each). **Simplify whenever possible.**

a. $r(x) = (\sqrt[3]{x} - e^{2x})(\sqrt[3]{x} + e^{2x})$ $= (x^{1/3} - e^{2x})(x^{1/3} + e^{2x}) = x^{2/3} - e^{4x}$

$$\frac{dr}{dx} = \frac{2}{3}x^{-1/3} - 4e^{4x}$$

b. $f(x) = \dfrac{3x^3}{\cot(2x)}$ $= 3x^3 \tan(2x)$

$$\frac{df}{dx} = 9x^2 \tan(2x) + 3x^3 \sec^2(2x) \cdot 2$$

$$= 3x^2 \left\{ 3\tan(2x) + 2x\sec^2(2x) \right\}$$

c. $r(t) = 3e^{3t}\sin^3(\pi t)$

$$\frac{dr}{dt} = 9e^{3t}\sin^3(\pi t) + 9\pi e^{3t}\cos(\pi t)$$

$$= 9e^{3t}\left\{ \sin^3(\pi t) + \pi\cos(\pi t) \right\}$$

d. $h(x) = \dfrac{x\cos(x)}{x^3 + 3}$

$$\frac{dh}{dt} = \frac{\left[\cos(x) - x\sin(x)\right]\left[x^3+3\right] - x\cos(x)(3x^2)}{(x^3+3)^2}$$

Problem #6 (continued)

e. $s(\theta) = e^{2\theta}\sin(\theta e^{\theta})$

$$\frac{ds(\theta)}{d\theta} = 2e^{2\theta}\sin(\theta e^{\theta}) + e^{2\theta}\cos(\theta e^{\theta})[e^{\theta} + \theta e^{\theta}]$$

$$= e^{2\theta}\left[2\sin(\theta e^{\theta}) + (e^{\theta} + \theta e^{\theta})\cos(\theta e^{\theta})\right]$$

7. (9 points) Find the **second** derivative of $g(x) = \dfrac{3x^{5/2} + 8x^{3/4}}{\sqrt{x}}$. **Your answer MUST be simplified.**

$$g(x) = 3x^{\frac{5}{2} - \frac{1}{2}} + 8x^{3/4 - 1/2}$$

$$= 3x^{2} + 8x^{1/4}$$

$$g'(x) = 6x + 2x^{-3/4}$$

$$g''(x) = 6 - \frac{3}{2}x^{-7/4}$$

8. (13 points) For the curve defined by $x^2y + y^3x = 2$, use implicit differentiation to find an expression for $\frac{dy}{dx}$. Using your formula for $\frac{dy}{dx}$, find the equation of the line (in slope-intercept form) tangent to the curve at the point $(1,1)$.

$$2xy + x^2\frac{dy}{dx} + 3y^2x\frac{dy}{dx} + y^3 = 0$$

$$\frac{dy}{dx}\left[x^2 + 3xy^2\right] = -2xy - y^3$$

$$\boxed{\frac{dy}{dx} = -\frac{2xy + y^3}{x^2 + 3xy^2}} \qquad \left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{2+1}{1+3} = -\frac{3}{4}$$

$$y_T = -\frac{3}{4}x + b$$

$$1 = -\frac{3}{4}\cdot 1 + b \Rightarrow b = 7/4$$

$$\boxed{y_T = -\frac{3}{4}x + \frac{7}{4}}$$

BONUS QUESTION (10 Points) If $f(x) = (2x^2 - 9x + 6)e^x$, determine the value (or values) of x where the tangent to the graph of $f(x)$ is horizontal.

$$\frac{df}{dx} = (4x - 9)e^x + (2x^2 - 9x + 6)e^x$$

$$= (2x^2 - 5x - 3)e^x \Rightarrow x = \frac{5 \pm \sqrt{25+24}}{4}$$

$$= \frac{5}{4} \pm \frac{1}{4}7$$

$$= 3, -1/2$$

$$\boxed{f'(x) = 0 \text{ @ } x = -\tfrac{1}{2} \text{ \& } 3}$$