# Finding the Derivative of the Inverse of a Function Whose Derivative is Known

**We have developed derivatives for numerous functions such as the six primary trigonometric functions and the exponential function. Each of these functions has a corresponding inverse function and we need a procedure for differentiating them.**

**We will develop a generic procedure for producing the desired derivatives and then apply it to several trigonometric functions and the exponential functions.**

**Assume the inverse of the function** $f(x)$ **is denoted by** $y = f^{-1}(x)$ **and the derivative of** $f(x)$ **is known and denoted by** $f'(x)$**. We seek to find the derivative of** $f^{-1}(x)$**, or** $\frac{dy}{dx}$**.**

**Starting with** $y = f^{-1}(x)$ **and applying** $f(x)$ **to both sides yields**

$$f(y) = f(f^{-1}(x)) = x \qquad \textbf{(1)}$$

**based on the properties of inverses. Differentiating Eq. 1 using implicit differentiation yields**

$$f'(y)\frac{dy}{dx} = 1 \quad \textbf{and, therefore,} \quad \frac{dy}{dx} = \frac{1}{f'(y)}.$$

**Substituting** $f^{-1}(x)$ **for** $y$ **yields**

$$\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}. \qquad \textbf{(2)}$$

**To the uninitiated, this looks like just a jumble of symbols, so let's apply it to an actual function. There are two ways to do this. First, we can mimic the process carried out above or, second, we can just apply the formula. Let's do both for the function** $\sin(x)$**.**

**Let** $y = \sin^{-1}(x)$**. This implies that** $\sin(y) = \sin(\sin^{-1}(x)) = x$**. Differentiating both sides using implicit differentiation yields**

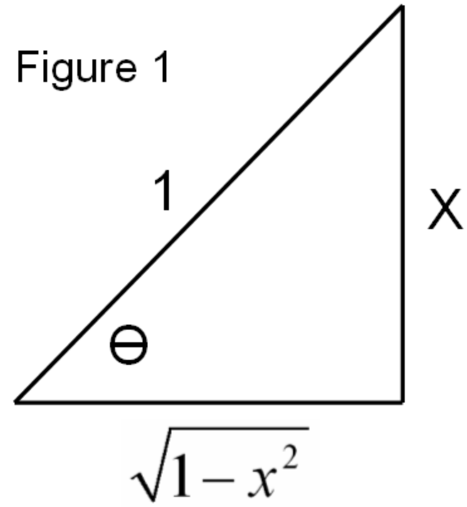
$$\cos(y)\frac{dy}{dx} = 1 \quad \textbf{and} \quad \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\cos(\sin^{-1}(x))} \qquad \textbf{(3)}$$

**How do we make sense of** $\cos(\sin^{-1}(x))$**? What we want is the cosine of an angle whose sine is x. So, let's draw a right triangle with an angle whose sine is x and then determine the cosine of that angle.**

**Figure 1 illustrates an angle** $\theta$ **whose sine is** $x$**. Using the Pythagorean Theorem, we can fill in the missing side of the triangle and determine that the cosine of** $\theta$ **is**

$\cos(\theta) = \sqrt{1-x^2}$ **. Therefore** $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$ **and we now know that the derivative of** $\sin^{-1}(x)$ **is**

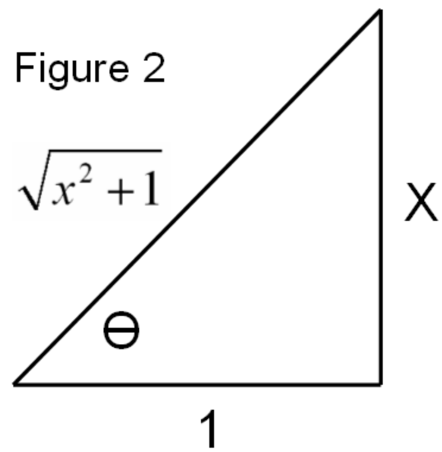
$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-x^2}} \qquad \textbf{(4)}$$




**If we look at the middle term in Eq. 4 we can see the elements of Eq. 2. Specifically the** $\text{"}\cos(\_)\text{"}$ **replaces** $f'(\_\_)$ **(since cos is the derivative of sin) and the argument of** $f'(\_\_)$ **is** $\sin^{-1}(x)$ **since that is the inverse function,** $f^{-1}(x)$**, we are dealing with.**

**Let's try this with** $\tan(x)$**. First, we need the derivative of** $\tan(x)$ **which is** $\sec^2(x)$**. Therefore, using Eq. 2 we can write the derivative of arctan(x) as follows:**

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{\sec^2(\tan^{-1}(x))}.$$




**Drawing a triangle appropriate to** $\theta = \tan^{-1}(x)$ **(Figure 2) and then filling in the missing side, this time the hypotenuse, yields an expression for** $\sec(\theta) = \sec(\tan^{-1}(x)) = \sqrt{x^2+1}$**. Squaring this expression and placing it in the denominator yields**

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{\sec^2(\tan^{-1}(x))} = \frac{1}{x^2+1}. \qquad \textbf{(5)}$$

**The same process works for the other four trigonometric functions. Let's try it for the inverse of the exponential function,** $e^x$**, namely,** $y = \ln(x)$**.**

**Exponentiating both sides we get** $e^y = e^{\ln(x)} = x$**. Differentiating this expression with respect to x using implicit differentiation yields** $e^y \frac{dy}{dx} = 1$ **and solving this expression for** $\frac{dy}{dx}$ **leads to** $\frac{dy}{dx} = \frac{1}{e^y}$**.**

**But** $e^y = x$ **so we can conclude that**

$$\frac{dy}{dx} = \frac{d}{dx}\ln(x) = \frac{1}{x} \qquad \textbf{(6)}$$

**The derivative of** $\log_a(x)$ **can then be derived from Eq. 6 using the change of base formula, namely** $\log_a(x) = \ln(x)/\ln(a)$ **yielding**

$$\frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)}\frac{d}{dx}\ln(x) = \frac{1}{\ln(a)x} \qquad \textbf{(7)}$$