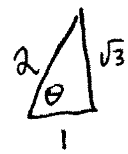
Calculus 1B Final Exam Spring 2010 Pg1 Name Solution Sec ________

Instructions: Calculators are **not allowed**. **No electronic devices may be used for any reason.** You may use an 8 ½ by 11 note sheet. Answers with little or no supporting work will receive little or no credit. Work must be **neat**, **organized** and **easily interpreted**. Graders may deduct if your work is not expressed in complete mathematical sentence (i.e., using verbs such as $=, \in$, and $\le$). **Your name and section number must be printed on ALL sheets.**

1. (2 Points each) Determine the values of x, y and z.

a. $x = \cos(\underbrace{\tan^{-1}(\sqrt{3})}_{\theta}) = \frac{1}{2}$



$\theta = \frac{\pi}{3}$

b. $y = e^{2\sin(\pi/6)-1} = e^{2\cdot\frac{1}{2}-1} = e^{1-1} = e^{0} = 1$

c. $z = \ln(e^{\sqrt{3}})/2 = \sqrt{3}/2$

2. (5 Points) Convert the following equation for x: $\ln(x)+\ln(x+16)-\ln(2x+5)=\ln(x+2)$ into an algebraic equation for x with no logarithms and then determine the value(s) of x that satisfy that equation.

$$\ln(x) + \ln(x+16) = \ln(x+2) + \ln(2x+5)$$

$$\ln\{x(x+16)\} = \ln\{(x+2)(2x+5)\}$$

$$x(x+16) = (x+2)(2x+5)$$

$$x^2 + 16x = 2x^2 + 9x + 10$$

$$x^2 - 7x + 10 = 0$$

$$(x-2)(x-5) = 0 \Rightarrow x = 2, 5$$

Both ok in original equation

3. (7 Points) Use logarithmic differentiation to determine the derivative of $f(x) = x^{\tan(x)}$

$$\ln(f(x)) = \tan(x)\ln(x)$$

$$\frac{1}{f(x)}\frac{df}{dx} = \sec^2(x)\ln(x) + \tan(x)\cdot\frac{1}{x}$$

$$\frac{df}{dx} = x^{\tan(x)}\left[\sec^2(x)\ln(x) + \frac{\tan(x)}{x}\right]$$

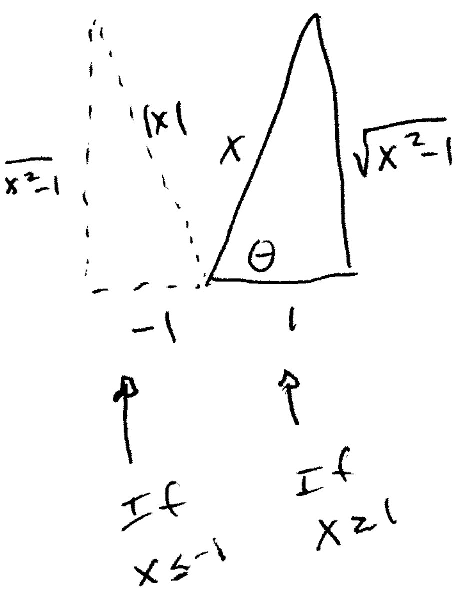
4. (10 Points) Using implicit differentiation, DERIVE the formula for the derivative of $y(x) = \sec^{-1}(x)$

$$\sec(y(x)) = x$$

$$\sec(y(x))\tan(y(x))\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec(\sec^{-1}(x))\tan(\underbrace{\sec^{-1}(x)}_{\theta})}$$

$$= \frac{1}{|x|\sqrt{x^2-1}}$$



Note:
$\tan(\theta) = \sqrt{x^2-1}$ if $x \ge 1$

$= -\sqrt{x^2-1}$ if $x \le -1$

"–" sign with "–" value for x

$\Rightarrow$ replace x with |x|

5. (3 Points each) Using the rules of differentiation, determine the derivatives of the following functions (Note: only 5c needs to be simplified):

a. $h(x) = \sin(\ln(x))$

$$\frac{dh}{dx} = \cos(\ln(x)) \cdot \frac{1}{x}$$

b. $g(x) = e^{-3x} \tan^{-1}(x)$

$$\frac{dg}{dx} = -3e^{-3x} \tan^{-1}(x) + e^{-3x} \frac{1}{1+x^2}$$

c. $h(x) = \dfrac{x^3 - 2x^{5/3} + 1}{\sqrt[3]{x}}$ $= \dfrac{x^{9/3} - 2x^{5/3} + x^{0/3}}{x^{1/3}} = x^{8/3} - 2x^{4/3} + x^{-1/3}$

$$\frac{dh}{dx} = \frac{8}{3}x^{5/3} - \frac{8}{3}x^{1/3} - \frac{1}{3}x^{-4/3} = \frac{8x^{5/3} - 8x^{1/3} - x^{4/3}}{3}$$

d. $s(t) = \dfrac{\sin(2t)}{t^3 - e^t}$

$$\frac{ds}{dt} = \frac{2\cos(2t)(t^3 - e^t) - \sin(2t)(3t^2 - e^t)}{(t^3 - e^t)^2}$$

6. (9 Points) The velocity of a particle is given by $v(t)=\frac{2}{t}+\cos(2\pi t)+4$ for $t\in[1,\infty)$. Given that the particle's initial position at t=1 is $x(1)=0$, find an equation for $x(t)$ for $t\geq 1$. Where is the particle at $t=2$ seconds?

absolute value can be dropped since $t\geq 1$

$$X(t)=2\ln(|t|)+\frac{1}{2\pi}\sin(2\pi t)+4t+C$$

$$X(1)=0+\frac{0}{2\pi}+4+C=0 \Rightarrow \boxed{C=-4}$$

$$\boxed{X(t)=2\ln(t)+\frac{1}{2\pi}\sin(2\pi t)+4t-4}$$

$$X(2)=2\ln(2)+0+8-4=4+2\ln(2)$$

7. (7 Points) The volume of a circular cone having base radius r and height h is given by the formula $V=\pi r^2 h/3$. At time t_0 the dimensions are r=6 cm and h=10 cm. At that time the radius is **increasing** at a rate of 0.5 cm per second while the height is remaining constant. What is the rate of change of the volume at time t_0? Be sure to indicate both the rate of increase and the relevant units.

$\boxed{\frac{dh}{dt}=0,\ h \text{ is a constant}}$

$$V(t)=\pi r^2(t)h/3$$

$$\left.\frac{dV}{dt}\right|_{t_0}=\left.\frac{\pi h}{3}2r(t)\frac{dr}{dt}\right|_{t_0}=\frac{\pi\cdot 10}{3}\cdot 2\cdot 6(0.5\text{ cm/sec})$$

$$=40\pi(0.5\text{ cm/sec})=20\pi\text{ cm/sec}$$

8. (15 Points total) Answer the following questions for the function $f(x) = x^2e^{-x}$

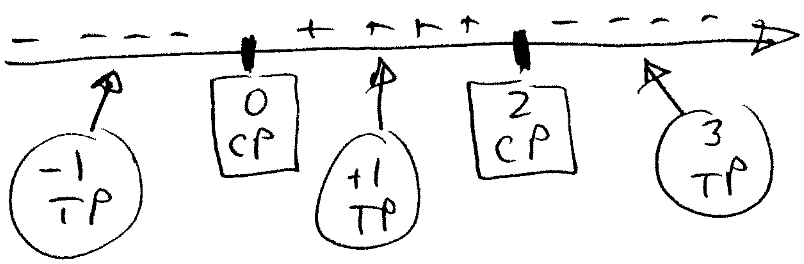
a. (5 Points) What are the critical points of $f(x)$ (only the x-values are required)?

$$\frac{df}{dx} = 2xe^{-x} + x^2e^{-x}(-1) = \underline{xe^{-x}(2-x)} = 0$$

Critical Pts: $x = 0$ or $x = 2$

Exists $\forall x$

b. (5 Points) Find the regions where $f(x)$ is increasing and regions where $f(x)$ is decreasing. Illustrate you answer with a number line sketch **and** then express your final answer in interval notation.



TP	SIGN $f'(x)$ = SIGN of $x(2-x)$
-1	$(-)(+) = (-)$
1	$(+)(+) = (+)$
3	$(+)(-) = (-)$

$f(x)$ increasing for $x \in (0, 2)$

$f(x)$ decreasing for $x \in (-\infty, 0) \cup (2, \infty)$

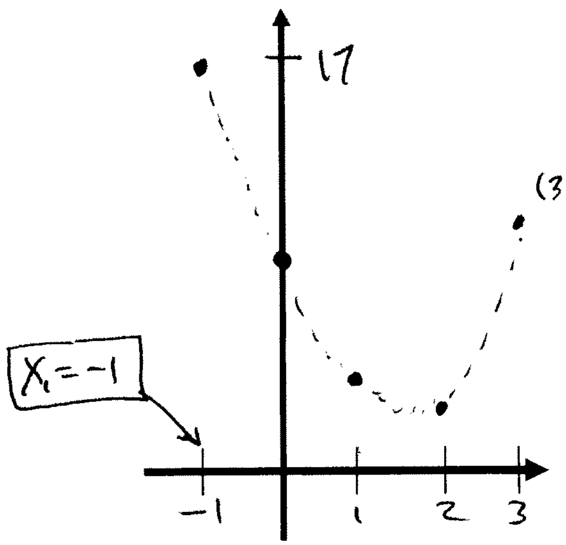
c. (5 Points) Identify any points where $f(x)$ achieves a local maximum or where it reaches a local minimum (provide <u>both</u> x and y coordinates). <u>Justify</u> why these points represent local extrema.

Local maximum @ $(2, 4/e^2)$ Increasing to the left, decreasing to the right

Local minimum @ $(0, 0)$ Decreasing on the left and increasing to the right

9. (12 Points) Approximate the area under the function $g(x) = x^3 - 8x + 10$ over the interval $x \in [-1, 3]$ using 4 equal subdivisions ($n = 4$) and the **LEFT**-endpoint rule. You are encouraged to draw the graph of $g(x)$ and use the table provided to organize your work.

$$\Delta x = \frac{3-(-1)}{4} = \frac{4}{4} = 1$$



$$\boxed{\hat{A}_{4\,LHR} = 32}$$

i	x_i	$g(x_i)$	ΔA_i
1	-1	$-1+8+10 = 17$	17
2	0	10	10
3	1	$1-8+10 = 3$	3
4	2	$8-16+10 = 2$	2

Sum = 32

Approximate Total Area= 32

10. (17 Points total) You are given a 10 foot string. You are instructed to cut the string into two pieces. With one piece you must construct a square (the square's circumference must be equal the length of that piece). With the other piece you must form an equilateral triangle (the triangle's circumference must equal the length of that piece). Where would you cut the original 10 foot string to minimize the combined area of the square and triangle. **Please** use X for the side of the square and Y for the side of the triangle.

a. (3 Points) What is the constraint equation?

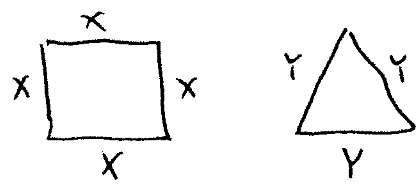
$$4X + 3y = 10$$

b. (4 Points) What is the objective function? [Hint: The area of the equilateral triangle is $\frac{\sqrt{3}}{4}Y^2$].

$$A(X,Y) = X^2 + \frac{\sqrt{3}}{4}Y^2$$

c. (10 Points) Determine the values of X and Y that minimize the total area? Be sure to demonstrate that your solution provides a minimum (not a maximum) total area.

$$y = \frac{10-4X}{3}$$



$$A(x) = X^2 + \frac{\sqrt{3}}{4}\left(\frac{10-4X}{3}\right)^2 = X^2 + \frac{\sqrt{3}}{36}(100 - 80x + 16x^2)$$

$$= \left(1 + \frac{4\sqrt{3}}{9}\right)X^2 - \frac{20\sqrt{3}}{9}X + \frac{25\sqrt{3}}{9}$$

$$\frac{dA}{dx} = 2\left(1 + \frac{4\sqrt{3}}{9}\right)X - \frac{20\sqrt{3}}{9} = 0$$

$$\boxed{\frac{d^2A}{dx^2} = 2\left(1+\frac{4\sqrt{3}}{9}\right) > 0 \Rightarrow CCUp \Rightarrow Min.}$$

$$+4\sqrt{3})X - 10\sqrt{3} = 0$$

$$\Rightarrow \boxed{X^* = \frac{10\sqrt{3}}{9+4\sqrt{3}}} \quad \boxed{y^* = \frac{10}{3} - \frac{4}{3}\left(\frac{10\sqrt{3}}{9+4\sqrt{3}}\right)}$$

Bonus Questions [10 points total] (Note: You may do **ONLY TWO** problems. **Provide solutions on Pg8**):

B.1 (5 Points) If $0 \le a \le 10$ and $0 \le b \le 10$, what is the maximum value of $M = a^2b^3$ if $a + b \le 10$?

B.2 (5 Points) Determine the value of $\lim_{x\to\infty}\left(x^{100}e^{-x}\right)$. NOTE: You must prove your answer.

B.3 (5 Points) Use Newton's Method with $x_0 = 2$ to estimate the solution of the equation $x^3 + x^2 = 7x + 3$ that is nearest to $x = 2$. You must produce a correct expression for x_2 (no need to simplify or evaluate).

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO!

Provide your solution to your selected bonus problems here:

Solution to First Bonus Problem [B. 1]

$a+b=10 \Rightarrow a=10-b$

$M=(10-b)^2 b^3=(100-20b+b^2)b^3=100b^3-20b^4+b^5$

$\frac{dM}{db}=300b^2-80b^3+5b^4 \quad \exists\ \forall b$

$=5b^2(60-16b+b^2)$

$b=0$ not a maximum

so $(b-10)(b-6)=0$

$b=10$ not a maximum

so $\boxed{b^*=6}$

$\frac{d^2M}{db^2}=600b-240b^2+20b^3<0$ @ $b=6 \Rightarrow$ MAX

$\boxed{a^*=4}$ $\boxed{M_{MAX}=4^2\cdot 6^3}$

Solution to Second Bonus Problem [B. 2]

$$\lim_{x\to\infty} x^{100}e^{-x}=\lim_{x\to\infty}\frac{x^{100}}{e^x}$$

$$\overset{\infty/\infty}{=}\lim_{x\to\infty}\frac{100x^{99}}{e^x}\overset{\infty/\infty}{=}\lim_{x\to\infty}\frac{99\cdot 100\,x^{98}}{e^x}\overset{\infty/\infty}{=} \text{etc. etc.}$$

$$=\lim_{x\to\infty}\frac{100!}{e^x}=\frac{\text{constant}}{\infty}=0$$

B.3 $f(x)=x^3+x^2-7x-3 \quad \frac{df}{dx}=3x^2+2x-7 \quad x_0=2$

$x_1=2-\frac{8+4-14-3}{12+4-7}=2-\frac{-5}{9}=\frac{23}{9}$

$$\boxed{x_2=\frac{23}{9}-\frac{\left(\frac{23}{9}\right)^3+\left(\frac{23}{9}\right)^2-7\left(\frac{23}{9}\right)-3}{3\left(\frac{23}{9}\right)^2+2\left(\frac{23}{9}\right)-7}}$$