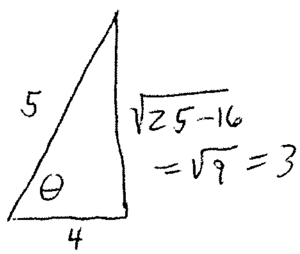
Calculus 1B Final Exam Summer 2012 Pg1 Name SOLUTION Sec ____

**Instructions:** Calculators and note sheets are **not allowed**. **No electronic devices may be used for any reason.** Answers with little or no supporting work will receive little or no credit. Work must be **neat**, **organized** and **easily interpreted**. Graders may deduct if your work is not expressed in complete mathematical sentence (i.e., using verbs such as $=, \in$, and $\leq$).

1. (4 points each) Determine the values of x and y.

a. $x = \log_2(4^{\log_2(8)}) = \log_2(4^3) = \log_2(2^6) = 6$

b. $y = \sin[\cos^{-1}(4/5)] = \sin(\theta) = 3/5$




2. (22 points total) **Differentiate each function**. Simplification is not required. Where indicated, state the domain of the original function. Each derivative is 4 Points. Each requested domain is 1 Point.

a. $f(x) = \tan^{-1}(e^{5x})$ Domain of $f(x)$: $(-\infty, \infty)$

$$\frac{df}{dx} = \frac{1}{1+e^{10x}} 5 e^{5x}$$

b. $g(x) = \cos(2x)\sinh(x^2)$

$$\frac{dq}{dx} = -2\sin(2x)\sinh(x^2) + \cos(2x)\cosh(x^2)\,2x$$

(See Over)

Problem #2 [Continued]

c. $s(x) = \dfrac{x\ln(x)}{x+e^x}$

$$\frac{ds}{dx} = \frac{(\ln(x)+1)(x+e^x) - x\ln(x)[1+e^x]}{(x+e^x)^2}$$

d. $h(x) = \ln(x^2 - 7x + 12)$ Domain of $h(x)$: $(-\infty, 3) \cup (4, \infty)$

$$x^2 - 7x + 12 > 0$$
$$(x-3)(x-4) > 0 \Rightarrow x < 3 \ \underline{\underline{\text{or}}}\ x > 4$$

$$\frac{dh}{dx} = \frac{1}{x^2 - 7x + 12}(2x - 7)$$

e. $s(t) = t^{\sin(t)}$

$$\ln(s(t)) = \sin(t)\ln(t)$$

$$\frac{1}{s(t)}\frac{ds}{dt} = \cos(t)\ln(t) + \sin(t)\frac{1}{t}$$

$$\frac{ds}{dt} = t^{\sin(t)}\left[\cos(t)\ln(t) + \frac{\sin(t)}{t}\right]$$

3. (15 Points total) Answer the following questions for the function $f(x) = \dfrac{x}{x^2+4}$

a. (1 point) Does $f(x)$ have any vertical asymptotes? No If so, for what x-value(s)? —

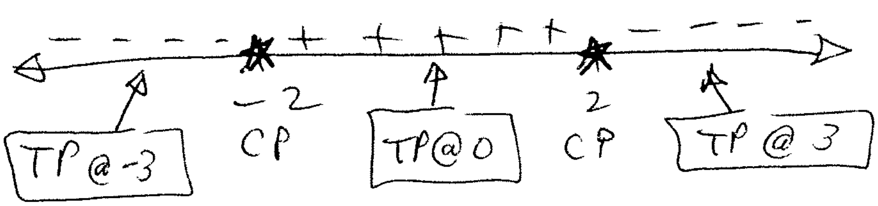
b. (2 points) Does $f(x)$ have a horizontal asymptote? Yes If so, what value does $f(x)$ approach as x approaches $-\infty$ ?: 0 As x approaches $+\infty$ ?: 0

c. (4 points) What are the critical points of $f(x)$ (only the x-values are required)?

$$\frac{df}{dx} = \frac{1(x^2+4) - x(2x)}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2} = \frac{(2-x)(2+x)}{(x^2+4)^2}$$

$\frac{df}{dx} = 0$ when $\boxed{x=2 \text{ or } x=-2}$ which are the x-values of the critical points of $f(x)$

d. (6 Points) Find the regions where $f(x)$ is increasing and regions where $f(x)$ is decreasing. Illustrate you answer with a number line sketch **and** then express your final answer in interval notation.




| TP | SIGN $f'(x)$ = sign of $(2-x)(2+x)$ |
|---|---|
| -3 | $\oplus \ominus = \ominus$ |
| 0 | $\oplus \oplus = \oplus$ |
| 3 | $\ominus \oplus = \ominus$ |

$f(x)$ decreasing for
$x \in (-\infty, -2) \cup (2, \infty)$

$f(x)$ increasing for
$x \in (-2, 2)$

e. (2 Points) Identify any points where $f(x)$ achieves a local maximum or where it reaches a local minimum (provide both x and y coordinates).

Local Maximum @ $(2, 1/4)$

Local Minimum @ $(-2, -1/4)$

(See Over)

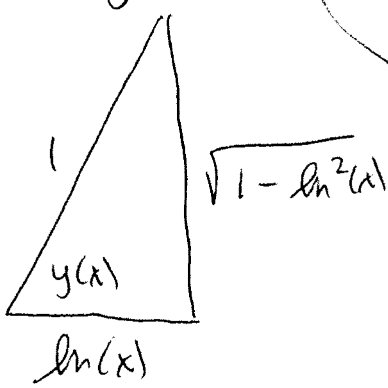
4. (12 Points) Using implicit differentiation, DERIVE the formula for the derivative of $y(x) = \cos^{-1}(\ln(x))$

$$\cos(y(x)) = \ln(x) \qquad \frac{1}{e} \le x \le e$$

$$-\sin(y(x))\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{-1}{x\sin(y(x))} = \frac{-1}{x\sqrt{1-\ln^2(x)}}$$

Decoding:



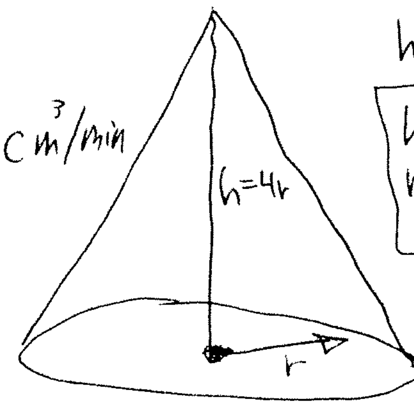

for $\frac{1}{e} < x < e$

so $\sin(y(x)) = \sqrt{1-\ln^2(x)}$

5. (10 Points) The height of the cone in this problem is always equal to 4 times its radius. At time $t_0$ the cone's height is $40cm$ and the cone's volume is increasing at a rate of $40\pi cm^3$ per minute. What is the rate of change of the cone's radius, $\frac{dr}{dt}$? Start by drawing a neat diagram showing all the given information. Make sure your answer includes units.

$h = 4r$

$$\frac{dV}{dt} = 40\pi \ cm^3/min \quad @ \ t_0$$




$$\boxed{\begin{array}{l} h = 40 \\ r = 10 \\ @ \ t_0 \end{array}}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi r^2 (4r) = \frac{4\pi}{3} r^3$$

$$\frac{dV}{d} = 4\pi r^2 \frac{dr}{dt}$$

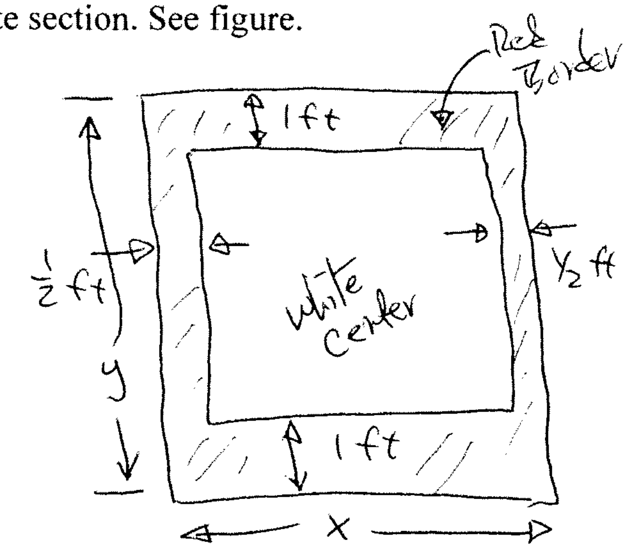
$$@ \ t = t_0 \quad \frac{dV}{dt} = 40\pi \ cm^3/min = 4\pi r^2 \frac{dr}{dt}$$

$$= 4\pi (10)^2 \frac{dr}{dt}$$

Therefore $$\left.\frac{dr}{dt}\right|_{t_0} = \frac{40\pi}{400\pi} = \frac{1}{10} \ cm/min$$

(See Over)

6. (14 Points total) A rectangular banner has a red border and a rectangular white center. The width of the border at the top and bottom is 1 foot. The border on each side is $\frac{1}{2}$ foot. The total area of the banner is $32\,ft^2$. Your job is to determine the dimensions of the banner, x and y, that will maximize the area of the interior white section. See figure.




a. (3 Points) What is the constraint equation?

$$xy = 32$$

b. (3 Points) What is the objective function?

$$W = (x-1)(y-2) \quad (\text{MAXIMIZE})$$

(Area of white center)

c. (8 Points) Determine the dimensions x and y that maximize the area of the white section. Be sure to demonstrate that your solution provides a maximum. What is the maximum area obtained?

$$y = 32/x \Rightarrow W(x) = (x-1)\left(\frac{32}{x} - 2\right)$$

$$W(x) = 32 - 2x - \frac{32}{x} + 2$$

$$\frac{dW}{dx} = -2 + 32x^{-2}$$

($x=0$ is Crit Pt but not a maximum)

$$-2 + \frac{32}{x^2} = 0$$

$$-2x^2 + 32 = 0 \qquad x^2 = 16 \qquad \boxed{x = 4} \text{ (not } -4)$$

$$\boxed{y = 8}$$

$$\frac{d^2W}{dx^2} = -64x^{-3} < 0 \Rightarrow \text{MAXIMUM} \qquad \boxed{W_{MAX} = 18}$$

for $x=4$

7. (12 Points) The velocity of a particle is given by $v(t) = 4\sinh(2t) + 5t^2$ for $t \in [0,10]$.

a. (8 points) Given that the particle's initial position at $t = 0$ is $x(0) = 3$, find an equation for $x(t)$ for $t \in [0,10]$.

$$x(t) = 2\cosh(2t) + \frac{5}{3}t^3 + C$$

$$x(0) = 3 = 2 + 0 + C \Rightarrow C = 1$$

Therefore $\boxed{x(t) = 2\cosh(2t) + \frac{5}{3}t^3 + 1}$

b. (4 points) What is the particle's position at $t = 3$ seconds? Express $x(3)$, if possible, using exponential functions. Can you estimate the numeric value of $x(3)$?

$$x(3) = 2\cosh(6) + 46$$

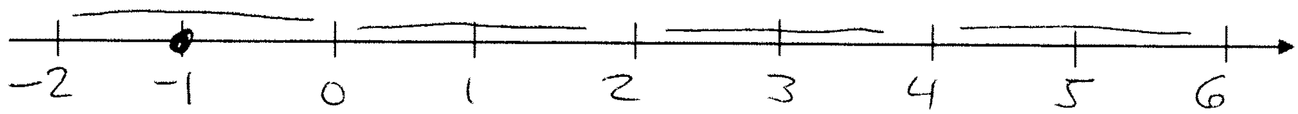
$$= 2\,\frac{e^6 + e^{-6}}{2} + 46$$

$$= e^6 + e^{-6} + 46$$

$e^3 \approx 20$ so $e^6 \approx 400$ & $e^{-6}$ is negligible

Therefore $x(3) \approx 446$

8. (7 Points) Approximate the area under the function $g(x) = 2x^2 - 7x + 3$ over the interval $x \in [-2, 6]$ using **4 equal subdivisions** (n=4) and the **midpoint** rule. Please use the table provided to organize your calculation of the estimated area. Use of the provided number line may be helpful but is optional.




$\Delta x =$

| $i$ | $x_i$ | $g(x_i)$ | $\Delta A_i$ |
|---|---|---|---|
| 1 | -1 | 2+7+3 = 12 | 24 |
| 2 | 1 | 2-7+3 = -2 | -4 |
| 3 | 3 | 18-21+3 = 0 | 0 |
| 4 | 5 | 50 -35+3 = 18 | 36 |

**Approximate Total Area=** 56

**Bonus Questions [5 points each]** (Note: You may do **ONLY TWO** problems. Provide your answers on page 9.)

B.1 (5 Points) If a and b are both greater than zero and $a^2 + b^2 = 8$, what values of a and b maximize $Q = a^2 b$ ?

B.2 (5 Points) Determine the value of $\lim_{x \to 0} \frac{\tan(x)}{\sinh(x)}$.

B.3 (5 Points) Using implicit differentiation, derive the derivative of $y(x) = \sinh^{-1}(x)$

B.4 (5 Points) Use the definite integral $\int_{-2}^{6} (2x^2 - 7x + 3)dx$ to determine the exact area for problem 8.

**Provide your solution to your selected bonus problems here:**

**Solution to First Chosen Bonus Problem [B. 1 ]**

$$a^2+b^2=8 \Rightarrow a^2=8-b^2 \Rightarrow Q=(8-b^2)b = 8b-b^3$$

$$\frac{dQ}{db} = 8-3b^2=0 \Rightarrow b=\sqrt{\frac{8}{3}} = \frac{2}{3}\sqrt{6}$$

$$\frac{d^2Q}{db^2} = -6b<0 \quad \forall b>0$$

$$\boxed{\begin{array}{l} b^* = \frac{2}{3}\sqrt{6} \\ a^* = \frac{4}{3}\sqrt{3} \end{array}}$$

B.2 $\lim_{x\to 0} \frac{\tan(x)}{\sinh(x)} \overset{0/0}{=} \lim_{x\to 0} \frac{\sec^2(x)}{\cosh(x)} = \frac{1^2}{1} = 1$

**Solution to Second Chosen Bonus Problem [B. 3 ]**

$$\sinh(y(x)) = x, \quad \cosh(y(x))\frac{dy}{dx} = 1, \quad \underbrace{\frac{dy}{dx} = \frac{1}{\cos(\sinh^{-1}(x))}}$$

This answer is sufficient for full credit

Decoding for this problem

but $\cosh^2(\sinh^{-1}(x)) - \sinh^2(\sinh^{-1}(x)) = 1$

so $\cosh^2(\sinh^{-1}(x)) = 1+x^2$

$\cosh(\sinh^{-1}(x)) = \sqrt{1+x^2}$

so $\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}}$

B.4 $\int_{-2}^{6}(2x^2-7x+3)\,dx = \left(\frac{2}{3}x^3 - \frac{7}{2}x^2+3x\right)\Big|_{-2}^{6} = \frac{183}{3}$ (after a little arithmetic!)

**DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO!**

**Note: You may use this page as a scratch area … It will NOT be graded.**