1. Local Behavior of Inhomogeneous Linear Equations

In this section we explore methods for determining the local behavior of solutions to an inhomogeneous differential equation. There are many ways to find the local behavior of a particular solution. Here we apply the method of dominant balance to the differential equation.

Method of dominant balance

We encountered the technique of dominant balance in the previous lecture (Lecture 10), which consists of three steps as follows:

I. Drop all terms that appear small and replace the exact equation by an asymptotic relation.

II. Replace the asymptotic relation with an equation by exchanging the \( \sim \) sign for an = sign and solve the resulting equation.

III. Check whether the solution is consistent with the approximation made in (I). If it is consistent, the controlling factor (and not the leading behavior) obtained from the dominant balance relation is the same as that the exact solution.

There are several cases to consider. First, we suppose that \( x_0 \) is an ordinary point of the associated homogeneous differential equation and a point of analyticity of the inhomogeneity. In this case the general solution has a Taylor series representation.

Example 1. Taylor series representation of the general solution. What is the local behavior of the general solution to

\[
y' + xy = x^3
\]

at \( x = 0 \)?

Solution:

Step 1: \( x = 0 \) is an ordinary point of the homogeneous equation \( y' + xy = 0 \) and a point at which \( x^3 \) is analytic.

Step 2: Taylor series representation of \( y(x) \):

\[
y(x) = \sum_{n=0}^{\infty} a_n x^n
\]

Step 3: Substituting (2) into (1) and equating coefficients of like powers of \( x \) gives a recursion relation for \( a_n \):

\[
a_1 = 0
\]

\[
n a_n + a_{n-2} = \begin{cases} 
0 & n \geq 2, n \neq 4 \\
1 & n = 4 
\end{cases}
\]
Step 4: (3) and (4) gives
\[ a_2 = -\frac{a_0}{2}, \]
\[ a_4 = \frac{1}{4} + \frac{a_0}{8}, \]
\[ a_n = \left( -\frac{1}{n} \right) a_{n-2} \]
\[ = \left( -\frac{1}{n} \right) \left( -\frac{1}{n-2} \right) a_{n-4} \]
\[ = \left( -\frac{1}{n} \right) \left( -\frac{1}{n-2} \right) \left( -\frac{1}{n-4} \right) a_{n-6} \]
\[ \Rightarrow \]
\[ = \left( -\frac{1}{n} \right) \left( -\frac{1}{n-2} \right) \left( -\frac{1}{n-4} \right) \ldots \left( -\frac{1}{n-(n-6)} \right) a_{n-(n-4)} \]
\[ = \left( -\frac{1}{n} \right) \left( -\frac{1}{n-2} \right) \left( -\frac{1}{n-4} \right) \ldots \left( -\frac{1}{6} \right) a_4 \]
\[ = \left( -\frac{1}{n} \right) \left( -\frac{1}{n-2} \right) \left( -\frac{1}{n-4} \right) \ldots \left( -\frac{1}{6} \right) a_4 \]
\[ = \left( -\frac{1}{2} \right) \left( \frac{1}{n/2} \right) \left( \frac{1}{(n-2)/2} \right) \ldots \left( \frac{1}{3} \right) a_4 \]
\[ = -\left( -\frac{1}{2} \right) ^n \frac{1}{(n/2)!} a_4 \]
\[ = -\left( -\frac{1}{2} \right) ^n \frac{1}{(n/2)!} \left( \frac{1}{4} + \frac{a_0}{8} \right) \]

Step 5: So we have all \( a_n \) as

\[ a_2 = -\frac{a_0}{2}, \]
\[ a_{2n} = -\left( -\frac{1}{2} \right) ^{n-3} \frac{1}{n!} \left( \frac{1}{4} + \frac{a_0}{8} \right) \quad n \geq 2 \]
\[ a_{2n+1} = 0 \quad n \geq 0 \]

(5)

This completes our determination of the local behavior of \( y(x) \).

If \( x_0 \) is an ordinary point of the differential equation but it is not a point of analyticity of the inhomogeneity, then, although all solutions to the homogeneous equation can be expanded in Taylor series, a particular solution to the inhomogeneous equation does not have a Taylor series expansion. In this case we use the method of dominant balance to find the behavior of a particular solution.

Example 2. Local behavior of solutions at an ordinary point of the associated homogeneous equation where the inhomogeneity is not analytic. Let us find the leading behavior of a particular solution to

\[ y' + xy = \frac{1}{x^4} \]

near \( x = 0 \).

Step 1: There are three dominant balances to consider:

(a) \( y' \sim xy, \ x^{-4} \ll xy \quad (x \to 0) \). The solution of this asymptotic differential equation is \( y \sim ae^{-x^2/2} \sim 0 \quad (x \to 0), \) which is not consistent with the condition that \( x^{-4} \ll xy \quad (x \to 0) \).
(b) \( xy \sim x^{-4}, \ y' \ll x^{-4} \quad (x \to 0) \). This asymptotic relation implies that 
\( y \sim x^{-5} \quad (x \to 0), \) which violates the condition that \( y' \ll x^{-4} \quad (x \to 0). \) This is also inconsistent.

(c) \( y' \sim x^{-4}, \ xy \ll x^{-4} \quad (x \to 0). \) This solution to this asymptotic differential equation is \( y \sim \frac{1}{3} x^{-3} \quad (x \to 0), \) which is consistent with the condition 
\( xy \ll x^{-4} \quad (x \to 0). \)

Step 2: So the only consistent leading behavior of \( y(x) \) is \( y(x) \sim -\frac{1}{3} x^{-3} \quad (x \to 0). \)
We set
\[
y(x) = -\frac{1}{3} x^{-3} + C(x)
\]
where
\[
C(x) \ll -\frac{1}{3} x^{-3}
\]
Step 3: Substituting (7) into (6) gives
\[
C' + xC = \frac{1}{3} x^{-2}
\]
Step 4: If we proceed as above using the method of dominant balance, we find that the leading behavior of \( C(x) \) as
\[
C(x) \sim -\frac{1}{3} x^{-1}
\]
Step 5: Continuing in this fashion, we obtain the full local behavior of \( y(x): \)
\[
y(x) \sim -\frac{1}{3} x^{-3} - \frac{1}{3} x^{-1} + a_0 + \frac{1}{3} x - \frac{1}{2} a_0 x^2 + \cdots, \quad x \to 0
\]
where \( a_0 \) is arbitrary, which is determined by the initial condition on the solution \( y(x). \)

Example 3. Verify the asymptotic solution of Example 2 using Matlab.
\[
\begin{align*}
y' + xy &= \frac{1}{x^4} & 0 < x \leq 1 \\
y(1) &= 0
\end{align*}
\]
Applying the initial condition, we can find \( a_0 \) of the asymptotic solution:
\[
a_0 = \frac{2}{3}
\]