Implicit Methods for Linear Homogeneous Transport Equation:

\[ u_t + u_x = 0 \]

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1st-Order BDF + Central Difference:

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_j^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = 0
\]

2nd-Order BDF2 + Central Difference:

\[
\frac{3u_j^{n+2} - (4u_j^{n+1} - u_j^n)}{\Delta t} + 2 \frac{u_j^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = 0
\]

2nd-Order TR-BDF2 + Central Difference:

Stage 1:

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{4} \left( \frac{u_j^{n+1} - u_{j-1}^{n+1}}{2\Delta x} + \frac{u_j^{n+1} - u_{j-1}^{n-1}}{2\Delta x} \right) = 0
\]

Stage 2:

\[
\frac{3u_j^{n+1} - (4u_j^n - u_j^n)}{\Delta t} + \frac{u_j^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = 0
\]

2nd-Order C-N + Central Difference:

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2} \left( \frac{u_j^{n+1} - u_{j-1}^{n+1}}{2\Delta x} + \frac{u_j^{n+1} - u_{j-1}^{n-1}}{2\Delta x} \right) = 0
\]

1st-Order BDF + 2nd-Order Upwind:

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{3u_j^{n+1} - 4u_j^{n-1} + u_{j+1}^{n+1}}{2\Delta x} = 0
\]

2nd-Order BDF2 + 2nd-Order Upwind:

\[
\frac{3u_j^{n+2} - (4u_j^{n+1} - u_j^n)}{\Delta t} + 2 \frac{3u_j^{n+2} - 4u_j^{n+1} + u_{j-2}^{n+2}}{2\Delta x} = 0
\]

2nd-Order TR-BDF2 + 2nd-Order Upwind:

Stage 1:

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{4} \left( \frac{3u_j^{n+1} - 4u_j^{n-1} + u_{j-1}^{n-2}}{2\Delta x} + \frac{3u_j^{n+1} - 4u_j^{n-1} + u_{j-2}^{n-2}}{2\Delta x} \right) = 0
\]

Stage 2:

\[
\frac{3u_j^{n+1} - (4u_j^n - u_j^n)}{\Delta t} + \frac{3u_j^{n+1} - 4u_j^{n-1} + u_{j-2}^{n+1}}{2\Delta x} = 0
\]

2nd-Order C-N + 2nd-Order Upwind:

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2} \left( \frac{3u_j^{n+1} - 4u_j^{n-1} + u_{j-1}^{n+1}}{2\Delta x} + \frac{3u_j^{n+1} - 4u_j^{n-1} + u_{j-2}^{n+1}}{2\Delta x} \right) = 0
\]