Area and Estimating with Finite Sums

Part 1
Problem:

Find the area of a region $R$ bounded below by the $x$-axis, on the sides by the lines $x = a$ and $x = b$, and above by a curve $y = f(x)$, where $f$ is continuous on $[a, b]$ and $f(x) \geq 0$ for all $x$ in $[a, b]$. 
First Step: Subdivide $[a, b]$ into $n$ subintervals of equal width of $\Delta x$. 

$y = f(x)$

1st subinterval, 2nd subinterval, k-th subinterval, (n-1)-th subinterval, n-th subinterval
Second Step: On each subinterval, draw a rectangle to approximate the area of the curve over the subinterval.
Third Step: Determine the area of each rectangle

- Width of the $k$-th rectangle $= \Delta x$
- Height of $k$-th rectangle $= f(c_k)$
- Area of $k$-th rectangle $= A_k = f(c_k) \cdot \Delta x$
Fourth Step: Sum up the areas of the rectangles to approximate the area under the curve

\[ A \approx A_1 + A_2 + A_3 + \cdots + A_n \]
\[ = f(c_1) \cdot \Delta x + f(c_2) \cdot \Delta x + f(c_3) \cdot \Delta x + \cdots + f(c_n) \cdot \Delta x \]
Upper sum method: \((c_k, f(c_k))\) is the absolute maximum of \(y = f(x)\) on the \(k\)-th subinterval
Lower sum method: \((c_k, f(c_k))\) is the absolute minimum of \(y = f(x)\) on the \(k\)-th subinterval.
Midpoint method: $c_k$ is the midpoint of the $k$-th subinterval
Right endpoint method: $c_k$ is the right endpoint of the $k$-th subinterval.
Left endpoint method: $c_k$ is the left endpoint of the $k$-th subinterval.
Use finite approximation to estimate the area under $f(x) = 3x + 1$ over $[2,6]$ using a lower sum with four rectangles of equal width.

Solution:

$$f(x) = 3x + 1$$
$$a = 2 \text{ and } b = 6$$
$$n = 4$$
Example 1 (continued)

\[ f(x) = 3x + 1 \text{ over } [2,6] \]

\[ \Delta x = \frac{\text{width of the interval}[a, b]}{\text{number of subintervals}} \]

\[ = \frac{b - a}{n} \]

\[ \Delta x = \frac{6 - 2}{4} = 1 \]

Note: in this example, the lower sum method is the same as the left endpoint method.
Example 1 (continued)
\[ f(x) = 3x + 1 \text{ over } [2,6] \]
Example 1 (continued)

\[ f(x) = 3x + 1 \text{ over } [2,6] \]

\[
A \approx f(c_1) \cdot \Delta x + f(c_2) \cdot \Delta x \\
+ f(c_3) \cdot \Delta x + f(c_4) \\
\cdot \Delta x
\]

\[
A \approx f(2) \cdot 1 + f(3) \cdot 1 + f(4) \\
\cdot 1 + f(5) \cdot 1
\]

\[
A \approx 7 + 10 + 13 + 16 = 46
\]

**Answer:** The area under the curve \( y = 3x + 1 \) over the interval \([2,6]\) is approximately 46.
Example 2

Using the midpoint rule, estimate the area under \( f(x) = \frac{1}{x} \) over \([1,9]\) using four rectangles.

Solution:

\[
\begin{align*}
    f(x) &= \frac{1}{x} \\
    a &= 1 \text{ and } b = 9 \\
    n &= 4
\end{align*}
\]
Example 2 (continued)

\[ f(x) = \frac{1}{x} \text{ over } [1, 9] \]

\[ \Delta x = \frac{\text{width of the interval } [a, b]}{\text{number of subintervals}} \]

\[ = \frac{b - a}{n} \]

\[ \Delta x = \frac{9 - 1}{4} = 2 \]
Example 2 (continued)

\[ f(x) = \frac{1}{x} \] over \([1,9]\)
Example 2 (continued)

\[ f(x) = \frac{1}{x} \] over \([1,9]\)

\[ A \approx f(c_1) \cdot \Delta x + f(c_2) \cdot \Delta x + f(c_3) \cdot \Delta x + f(c_4) \cdot \Delta x \]

\[ A \approx f(2) \cdot 2 + f(4) \cdot 2 + f(6) \cdot 2 + f(8) \cdot 2 \]

\[ A \approx \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 2 + \frac{1}{6} \cdot 2 + \frac{1}{8} \cdot 2 = \frac{25}{12} \]

Answer: The area under the curve \( y = \frac{1}{x} \) over the interval \([1,9]\) is approximately \( \frac{25}{12} \).
DO YOUR HOMEWORK.