Area and Estimating with Finite Sums

Part 3

Average Value of a Nonnegative Continuous Function
The average value of \( \{x_1, x_2, x_3, \ldots, x_n\} \) is

\[
\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}
\]

What is the average value of a continuous function \( f \) on an interval \([a, b]\)?
Average Value

• If $f$ is a constant value $c > 0$ on an interval $[a, b]$, then its average value should be $c$.

• Average value of $f = \frac{\text{Area under } f \text{ over } [a,b]}{\text{Width of interval}} = \frac{c(b-a)}{b-a} = c$
Average Value

Suppose $f$ is continuous on $[a, b]$, $f(x) \geq 0$ for all $x$ in $[a, b]$ and that $A$ is the area under the curve $y = f(x)$ over that interval. Then the average value of $f$ over $[a, b]$ is:

$$av(f) = \frac{A}{b-a}$$
Example

Use a finite sum to estimate the average value of $f(x) = \cos(x)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by partitioning the interval into four subintervals of equal length and using the lower sum method.

Solution:
First, we need to find the area under $f(x) = \cos(x)$ over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. 
Example (continued)

\[ f(x) = \cos(x) \]
\[ a = -\frac{\pi}{2} \text{ and } b = \frac{\pi}{2} \]
\[ n = 4 \]

\[ \Delta x = \frac{\text{width of the interval}}{\text{number of subintervals}} = \frac{b - a}{n} \]
\[ \Delta x = \frac{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)}{4} = \frac{\pi}{4} \]
Example (continued)

\[
A \approx f(c_1) \cdot \Delta x + f(c_2) \cdot \Delta x \\
+ f(c_3) \cdot \Delta x + f(c_4) \\
\cdot \Delta x
\]

\[
A \approx f\left(-\frac{\pi}{2}\right) \cdot \frac{\pi}{4} + f\left(-\frac{\pi}{4}\right) \cdot \frac{\pi}{4} \\
+ f\left(\frac{\pi}{4}\right) \cdot \frac{\pi}{4} + f\left(\frac{\pi}{2}\right) \cdot \frac{\pi}{4}
\]

\[
A \approx 0 \cdot \frac{\pi}{4} + \frac{\sqrt{2}}{2} \cdot \frac{\pi}{4} + \frac{\sqrt{2}}{2} \cdot \frac{\pi}{4} + 0
\]

\[
\frac{\pi}{4} = \frac{\sqrt{2} \cdot \pi}{4}
\]
Example (continued)

Now we can find the average value:

\[
\text{av}(f) = \frac{A}{b-a} \approx \frac{\sqrt{2} \cdot \pi}{4} = \frac{\sqrt{2}}{4}
\]

**Answer:** The average value of \( y = \cos(x) \) over the interval \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) is approximately \( \frac{\sqrt{2}}{4} \).
Girl: How do I look?

Boy: \[ \frac{\tan c}{\sin c} \]

Girl: Huh?

Boy: \[ \frac{\tan c}{\sin c} \]

\[ \begin{align*}
= & \left( \frac{\sin c}{\cos c} \right) \\
= & \frac{1}{\cos c} \\
= & \sec c
\end{align*} \]