Trigonometric Substitutions

Part 1:
Introduction and $\sqrt{a^2 - x^2}$
In order to evaluate integrals involving the above expressions, we will eliminate the radicals by using an appropriate trigonometric substitution.
Example

Evaluate

\[ \int \frac{dx}{x^2 \sqrt{4 - x^2}} \]

Solution:
This has an expression involving the form
\[ \sqrt{a^2 - x^2} \]
with \( a = 2 \).
We need $x$ to take on every value between $-a$ and $a$.

Since $\sin \theta$ takes on every value between -1 and 1 exactly once on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we will let

$$x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Now, since $1 - \sin^2 \theta = \cos^2 \theta$, this gives us

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a \sin \theta)^2}$$
Example (continued)

\[
\sqrt{a^2 - x^2} = \sqrt{a^2 - (a \sin \theta)^2} \\
= \sqrt{a^2 - a^2 \sin^2 \theta} \\
= \sqrt{a^2(1 - \sin^2 \theta)} \\
= \sqrt{a^2 \cos^2 \theta} \\
= |a \cos \theta|
\]

And since \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\), we know that \(\cos \theta \geq 0\), giving us

\[
\sqrt{a^2 - x^2} = a \cos \theta
\]
Example (continued)

\[ \int \frac{dx}{x^2 \sqrt{4 - x^2}} \]

\[ x = 2 \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \]

\[ dx = 2 \cos \theta \, d\theta \]

\[ \int \frac{dx}{x^2 \sqrt{4 - x^2}} = \int \frac{2 \cos \theta \, d\theta}{(2 \sin \theta)^2 \sqrt{4 - (2 \sin \theta)^2}} \]

\[ = \int \frac{2 \cos \theta \, d\theta}{4 \sin^2 \theta \cdot 2 \cos \theta} \]

J. Gonzalez-Zugasti, University of Massachusetts - Lowell
\[
\int \frac{2 \cos \theta \, d\theta}{4 \sin^2 \theta \cdot 2 \cos \theta} = \int \frac{d\theta}{4 \sin^2 \theta} \\
= \int \frac{1}{4} \csc^2 \theta \, d\theta \\
= \frac{1}{4} ( - \cot \theta ) + C = - \frac{1}{4} \cot \theta + C
\]

We want the answer in terms of \( x \)!
Example (continued)

\[ x = 2 \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \]

\[ \frac{x}{2} = \sin \theta = \frac{\text{OPP}}{\text{HYP}} \]

\[ \Rightarrow \sin^{-1} \left( \frac{x}{2} \right) = \theta \]

\[ \cot \theta = \frac{\text{ADJ}}{\text{OPP}} = \frac{\sqrt{4 - x^2}}{x} \]
Taking the results from our reference triangle, we get

\[
\int \frac{dx}{x^2\sqrt{4 - x^2}} = -\frac{1}{4}\cot\theta + C
\]

\[= -\frac{1}{4} \cdot \frac{\sqrt{4 - x^2}}{x} + C\]
“Real Fact” #812

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654,321