Integration of Rational Functions by Partial Fractions

Part 2:
Integrating Rational Functions
Rational Functions

Recall that a **rational function** is the quotient of two polynomials.

\[
\frac{1}{x} + \frac{3}{x - 1} + \frac{2}{x + 2}
\]

\[
= \frac{1(x - 1)(x + 2) + 3x(x + 2) + 2x(x - 1)}{x(x - 1)(x + 2)}
\]

\[
= \frac{6x^2 + 5x - 2}{x^3 + x^2 - 2x}
\]

The left side of this equation is easy to integrate. The right side is hard to integrate.
Suppose \( p(x) \) and \( q(x) \) are polynomials and

\[
f(x) = \frac{p(x)}{q(x)}.
\]

- If the degree of \( p(x) \) is less than the degree of \( q(x) \), then \( f(x) \) is a **proper** rational function.
- If the degree of \( p(x) \) is greater than or equal to the degree of \( q(x) \), then \( f(x) \) is an **improper** rational function.

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In theory, a polynomial with real coefficients can always be factored into a product of linear and quadratic factors. If a quadratic factor cannot be further decomposed into linear factors, then it is said to be irreducible. It can be proved that any proper rational function is expressible as a sum of terms (called partial fractions) having the form:

\[ \frac{A}{(ax + b)^k} \quad \text{or} \quad \frac{Bx + C}{(ax^2 + bx + c)^k} \]
Steps to Partial Fraction Decomposition

Suppose \( p(x) \) and \( q(x) \) are polynomials and

\[
f(x) = \frac{p(x)}{q(x)}.
\]

• Completely factor the denominator \( q(x) \) into linear and irreducible quadratic factors.

• Collect all repeated factors so that \( q(x) \) is expressed as a product of distinct factors of the form

\[
(ax + b)^m \quad \text{and} \quad (ax^2 + bx + c)^m
\]

where \((ax^2 + bx + c)^m\) is irreducible.
Steps to Partial Fraction Decomposition (continued)

• The structure of \( \frac{p(x)}{q(x)} \) is determined as follows:

**Linear Factors**

For each factor of the form \((ax + b)^m\), introduce the \(m\) terms

\[
\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_m}{(ax + b)^m}
\]

where \(A_1, A_2, \ldots, A_m\) are constants to be determined.
Irreducible Quadratic Factors

For each factor of the form \((ax^2 + bx + c)^m\), introduce the \(m\) terms

\[
\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}
\]

where \(A_1, A_2, \cdots, A_m, B_1, B_2, \cdots, B_m\) are constants to be determined.
Example 1

(1) \[ \frac{1}{(x-1)^2(x+3)^3(x^2+x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2} + \frac{E}{(x+3)^3} + \frac{F}{x^2 + x + 1} + \frac{G}{(x^2 + x + 1)^2} \]

(2) \[ \frac{5x+4}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 4} \]
Example 2

Evaluate

\[ \int \frac{1}{x^2 + x - 2} \, dx \]

Solution:

\[ \frac{1}{x^2 + x - 2} = \frac{1}{(x - 1)(x + 2)} \]
Example 2 (continued)

\[
\frac{1}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2}
\]

Now, multiply both sides of the equation by \((x - 1)(x + 2)\) to get

\[
1 = A(x + 2) + B(x - 1)
\]

To solve for \(A\) and \(B\), substitute values of \(x\) to make the various terms zero.
Example 2 (continued)

\[1 = A(x + 2) + B(x - 1)\]

Setting \(x = -2\) gives us:

\[1 = A(-2 + 2) + B(-2 - 1)\]

\[1 = -3B\]

\[-\frac{1}{3} = B\]

Setting \(x = 1\) gives us:

\[1 = A(1 + 2) + B(1 - 1)\]

\[1 = 3A\]

\[-\frac{1}{3} = A\]
An alternate way to solve for \( A \) and \( B \) is:

Take the equation

\[
1 = A(x + 2) + B(x - 1)
\]

Equate corresponding coefficients on both sides

\[
1 = Ax + 2A + Bx - B
\]

\[
0x + 1 = (A + B)x + (2A - B)
\]

\[
\begin{align*}
A + B &= 0 \\
2A - B &= 1
\end{align*}
\]

Solve the system of equations to get

\[
A = \frac{1}{3} \quad \text{and} \quad B = -\frac{1}{3}.
\]
Example 2 (continued)

Since

\[
\frac{1}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2}
\]

\[
= \frac{1/3}{x - 1} + \frac{-1/3}{x + 2}
\]

we can now say

\[
\int \frac{1}{(x - 1)(x + 2)} \, dx = \frac{1}{3} \int \frac{1}{x - 1} \, dx - \frac{1}{3} \int \frac{1}{x + 2} \, dx
\]

\[
= \frac{1}{3} \ln|x - 1| - \frac{1}{3} \ln|x + 2| + C
\]

\[
= \frac{1}{3} \ln \left| \frac{x - 1}{x + 2} \right| + C
\]
Example 3

Evaluate

\[ \int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} \, dx \]

Solution:
Since the degree of the numerator is larger than the degree of the denominator, this is an improper rational function!
When you have an improper rational function, the first thing you need to do is long division of polynomials to rewrite the improper rational function as the sum of a polynomial and a proper rational function.

Using long division of polynomials we get:

\[
\frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} = (3x^2 + 1) + \frac{1}{x^2 + x - 2}
\]
Example 3 (continued)

\[
\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} \, dx
\]

\[
= \int (3x^2 + 1) \, dx + \int \frac{1}{x^2 + x - 2} \, dx
\]

See Example 2

\[
= x^3 + x + \frac{1}{3} \ln \left| \frac{x - 1}{x + 2} \right| + C
\]
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