Solutions to Problems for S16.2

1 $$\frac{z^2+z+1}{2z+1-i\sqrt{3}}$$

Find the singularity: $2z+1-i\sqrt{3}=0$

$$2z=-1+i\sqrt{3}$$

$$z=\frac{-1+i\sqrt{3}}{2}$$

At $z=\frac{-1+i\sqrt{3}}{2}$, we have:

$$\frac{\left(\frac{-1+i\sqrt{3}}{2}\right)^2+\left(\frac{-1+i\sqrt{3}}{2}\right)+1}{0}=\frac{0}{0}$$

Use L'hôpital:

$$\lim_{z\to\left(\frac{-1+i\sqrt{3}}{2}\right)}\frac{z^2+z+1}{2z+1-i\sqrt{3}}=\lim_{z\to\left(\frac{-1+i\sqrt{3}}{2}\right)}\frac{2z+1}{2}$$

$$=\frac{2\left(\frac{-1+i\sqrt{3}}{2}\right)+1}{2}=\frac{i\sqrt{3}}{2}$$

$$\therefore\quad f(z)=\begin{cases}\dfrac{z^2+z+1}{2z+1-i\sqrt{3}} & z\neq\left(\dfrac{-1+i\sqrt{3}}{2}\right)\\[2ex] \dfrac{i\sqrt{3}}{2} & z=\left(\dfrac{-1+i\sqrt{3}}{2}\right)\end{cases}$$

2. $\dfrac{1}{z^4-1} = \dfrac{1}{(z^2+1)(z^2-1)} = \dfrac{1}{(z+1)(z-1)(z+i)(z-i)}$

expand in terms of Partial fractions

$$= \underbrace{\boxed{\frac{A}{z+1}}}_{a)} + \underbrace{\boxed{\frac{B}{z-1}}}_{b)} + \underbrace{\boxed{\frac{C}{z+i}}}_{c)} + \underbrace{\boxed{\frac{D}{z-i}}}_{d)}$$

a) already in L.S @ -1
for $0<|z+1|$
since finite # of terms in principal part; then
$z=-1$ is a simple pole

b) $\dfrac{B}{z-1} \Rightarrow$ already in the form of the L.S., about 1 in the region: $0<|z-1|$
so, since there is a finite number of terms in principal part of L.S.,
then $z=1$ is a simple pole.

c) $\dfrac{C}{z+i} \Rightarrow$ already in the form of L.S. about $-i$ in the region: $0<|z+i|$
so since there is a finite number of terms in the principal part of the L.S.

then $z=-i$ is a simple pole.

d) $\frac{D}{z-i} \Rightarrow$ already in the form of L.S. about i, in the region : $0<|z-i|$

so, since there is a finite number of terms in the principal part of the L.S, then i is a simple pole.

so, in summary: $1, -i, i, -1$ are all simple poles.

3. $$f(z) = \frac{2z+1-i\sqrt{3}}{(z^2+z+1)^2} = \frac{2\left(z+\left(\frac{1-i\sqrt{3}}{2}\right)\right)}{\left[\left(z-\left(\frac{-1+\sqrt{3}i}{2}\right)\right)\right]^2\left[z-\left(\frac{-1-\sqrt{3}i}{2}\right)\right]^2}$$

$$= \frac{2}{\left[z-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right]\left[z-\left(\frac{-1-\sqrt{3}i}{2}\right)\right]^2}$$

Expand in terms of partial fractions:

a) b) 4/

$$= \frac{A}{\left(z-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)} + \frac{B}{\left(z-\left(\frac{-1-\sqrt{3}i}{2}\right)\right)} + \frac{C}{\left(z-\left(\frac{-1-\sqrt{3}i}{2}\right)\right)^2}$$

a) $\Rightarrow \dfrac{A}{z-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}$ $\Rightarrow$ already in the form of the L.S. about $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ for

$0 < \left|z-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right|$

Since there is a finite number of terms in the principal part of the L.S, then $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ is a simple pole.

b) $\dfrac{B}{\left(z-\left(\frac{-1-\sqrt{3}i}{2}\right)\right)} + \dfrac{C}{\left(z-\left(\frac{-1-\sqrt{3}i}{2}\right)\right)^2}$ $\Rightarrow$ this is already in the form of the L.S about $\frac{-1-\sqrt{3}i}{2}$ in

the region: $0 < \left|z-\left(\frac{-1-\sqrt{3}i}{2}\right)\right|$

since there are a finite number of terms, then $\frac{-1-\sqrt{3}i}{2}$ is a pole of order 2

5

4. $\exp(1/z) = f(z)$

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for all } z$$

L.S @ 0 of $f(z)$:

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} \quad \text{for } 0<|z|<\infty$$

Expand:

$$\frac{1}{0!} + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

Since there are an infinite number of terms in the principal part of the series, $z=0$ is an essential singularity.