Lead Compensators Design Using Frequency Response Techniques

Ahmed Abu-Hajar, Ph.D.

**Frequency Response of Feedback System**

Transfer Function of Feedback System:

\[
T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)}
\]

\[
L(j\omega) = G(j\omega)H(j\omega)
\]

Frequency Response of Feedback System

\[
T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + L(j\omega)}
\]

Stability Considerations

Let assume that there is a frequency \( \omega_1 \) such that

\[
L(j\omega_1) = -1 = 180^\circ - 180^\circ
\]

\[
T(j\omega_1) = \frac{G(j\omega_1)}{1 + (-1)} = \frac{G(j\omega_1)}{1 - 1} = \infty
\]

Let \( \omega_{180} \) be the frequency at which the phase of \( L(j\omega) \) is \( 180^\circ \)

\[
L(j\omega_{180}) = 1 \quad L(j\omega_{180}) = \pm 180^\circ
\]

Important Notes:

1. Positive Feedback, system is not stable (oscillate at \( \omega_{180} \))

\[
|L(j\omega_{180})| > 1
\]

2. Infinite (or very high gain @ \( \omega_{180} \))

\[
|L(j\omega_{180})| = 1
\]

3. Stable and Tolerable Gain at \( \omega_{180} \)

\[
|L(j\omega_{180})| \ll 1
\]
Phase Margin & Gain Margin

Phase Margin (PM)
Let \( \omega_0 \) be the frequency at which the Open Loop Gain is 0dB (unity gain).

\[ PM = 180^\circ + \angle (\omega_0) \]

PM must be positive
\( PM > 30^\circ \text{ to } 60^\circ \).

Gain Margin (GM)
@\( \omega_{180} \) \( |L(j\omega_{180})| << 1 \text{ (0dB)} \).

\[ GM = -20 \log_{10} |L(j\omega_{180})| \]

GM must be positive
\( GM > 6 \text{dB} \).

Lead Compensator Design using Frequency response
The Transfer Function:
\[ G_c(z) = \frac{z + \alpha}{z - 1} \quad p > z > 0 \]
\[ G_p(z) = K_c \frac{z + \alpha}{z - 1} = K \frac{1 + \alpha}{1 + \frac{p}{\omega}} \]
\[ \alpha = \frac{z}{p} \quad 0 < \alpha < 1 \]

The Maximum Phase \( \varphi_m \)
\[ \sin (\varphi_m) = \frac{1 - \alpha}{1 + \alpha} \]
\[ \alpha = \frac{1 - \sin (\varphi_m)}{1 + \sin (\varphi_m)} \]

\( \omega_m \): The frequency @ which the phase is \( \varphi_m \)
\[ \omega_m = \frac{\pi}{\sqrt{\alpha}} \]

The gain is increased because of \( K_c \) and because of the zeros which gives+20dB/Dec
Designing Lead Compensators

Design Objective
1- Increase the phase at the frequency 0dB crossing, hence increases PM
2- Improves the velocity error constant $K_v$ by selecting the proper $K_c$

Steps:
1- Find the Loop Gain $L(s) = G(s)H(s)$.
2- Determine $K = \alpha K_c$ that would give the desired $K_v$.
3- Draw Bode Plot of $L_c(s) = KG(s)H(s)$.
   - Must compensate for $K$ changes the PM requirement.
   - Evaluate the PM.
4- Add $5^\circ - 12^\circ$ because of the increase in the gain due to the zero
5- Determine $\alpha$ from the required PM.
   - The new zero crossing occurs at frequency is where $\omega_m$ is located.
   - It is where the gain of $L_c(s)$ is given as:
   
   \[
   K_v = -20 \log \left( 1 \over \sqrt{\alpha} \right)
   \]
6- Determine the location of the zero and the pole as:
   \[
   a = \sqrt{\omega_m} \quad \beta = \frac{\omega_m}{\alpha} = \frac{2}{\alpha}
   \]
7- Choose the value of $K_c$ from $K = \alpha K_c$
8- Check the GM and PM requirements. If not met, reiterate the process.
   (Usually, you need to go to step 4 and add more phase).

Lead Compensator Example

\[
G_c(s) = \frac{4}{s(s + 2)}
\]

Design a lead compensator using Bode Plot Method such that
1- The velocity error constant $K_v = 20 \text{ sec}^{-1}$
2- Phase Margin PM = 50°
3- Gain Margin GM > 10.

Solution
\[
L(s) = G(s) = \frac{4}{s(s + 2)}
\]
\[
L_c(s) = K \frac{1 + \frac{2}{s}}{1 + \frac{2}{s}}
\]
\[
K_v = \lim_{s \to 0} sG_c(s)L(s)
\]
\[
K_v = \lim_{s \to 0} s \left( \frac{1 + \frac{2}{s}}{1 + \frac{2}{s}} \right) \left( \frac{4}{s(s + 2)} \right) = 20
\]
Lead Compensator Example

Draw Bode Plot of \( L_1(s) \)

\[
L_1(s) = \frac{20}{s(1 + \frac{1}{2})}
\]

Using approximated bode plot PM is found to be 17°.
Using Matlab, exact PM was found to be 17.9°.
We need to evaluate \( \phi_m \) of the compensator to get \( 50° + (5° - 12°) \)
The maximum phase of the compensator

\[
\phi_m = 50° - 17° + 5° = 38°
\]

Lead Compensator Example

Solve for \( \alpha \)

\[
\alpha = \frac{1 - \sin (\phi_m)}{1 + \sin (\phi_m)} = \frac{1 - \sin (38°)}{1 + \sin (38°)} = 0.239 \approx 0.24
\]

The gain \( (K_m) \) caused by the early zero

\[
K_m = -20 \log \left( \frac{1}{\sqrt{10}} \right) = -6.2 \text{ dB}
\]

The new zero crossing \( (\omega_m) \) is where the \( L_1(j\omega) = -6.2 \text{ dB} \)

From the Graph, that is located at \( \omega_m = 9 \text{ rad/sec} \)
Lead Compensator Example

\[ \alpha_m = 9 \]

The zero and the pole and \( K_c \) of the compensators are found as follows

\[ z = \sqrt{\frac{\omega_m}{\omega_n}} = 4.41 \quad p = \frac{\omega_m}{\alpha} = 18.4 \quad K_c = \frac{K}{\alpha} = 41.7 \]

The compensator is given as

\[ C_c(s) = \frac{41.7(s + 4.41)}{s + 18.4} = \frac{1}{10} \frac{s + 3.63}{1 + 18.4} \]
Lead Compensator Example

[Graph showing step response]

Lead Compensator Example

[Graph showing linear simulation results]
Lag Compensators Design Using Frequency Response Techniques

Ahmed Abu-Hajar, Ph.D.

Lag Compensator

The Transfer Function

\[ G_c(s) = \frac{s + z}{s + p} \quad : \omega > p \quad \beta = \frac{s}{p} \]
\[ G_c(z) = \frac{1 + \frac{s}{z}}{1 + \frac{z}{p}} \quad \beta > 1 \]

The Objective:

To provide attenuation (reduction in magnitude) after \( z \).

By reducing the magnitude before the 0dB crossing (\( \omega_0 \)), the phase margin improves.

\[ K_{an} = 20\log (\beta) \]

The location of \( z \) and \( p \) must be before \( \omega_0 \).

The attenuation is determined by the ratio \( \beta \)

Design Objectives:

Improving the steady state error while improving the PM.
Lag Compensator Example

Design a Lag Compensator using Frequency Response method such that:
1. The velocity error constant $K_v = 5 \text{ sec}^{-1}$
2. Phase Margin PM = 40°

Solution

The velocity error constant is given as:

$$K_v = \lim_{s \to \infty} s \left( K \frac{1 + \frac{s}{\omega_0}}{(1 + \frac{s}{\omega_0})} \right) \left( \frac{1}{s(1 + 2)} \right) = 5$$

$$K = 5$$

Lag Compensator Example

$K = 5$

K changes the 0dB crossing frequency $\omega_0$.

Investigate the effect of K on $\omega_0$.

$$L_1(s) = \frac{5}{s(1 + 2)}$$

Draw Bode Plot of $L_1(s)$

Uncompensated the PM is 30°
Due to $K = 5$, the PM is -12°
Not stable system.
Lag Compensator Example

Now, we need to locate the frequency at which we get a phase margin.
From the graph, \( \omega = 0.7 \), \( \text{PM} = 40^\circ \).
Add 12° correction factor, (acquired by trial and error).
\( \omega_0 = 0.5 \), \( \text{PM} = 52^\circ \).

Choose the 0dB crossing \( \omega_0 = 0.5 \)

\[
L_d(j\omega) = \frac{5}{j\omega (1 + j\omega)(1 + \frac{\omega_0^2}{2})}
\]

\[
|L_d(j\omega_0)| = \left| L_d(j0.5) \right| = \frac{5}{0.5(1 + 0.5^2)(1 + \frac{0.5^2}{2})} = 9.8 \approx 9 = 19\text{dB} \quad \beta \approx 9
\]

Choose \( z = 0.01 \) before \( \omega_0 \to p = 0.01 \)

\[
K_c = \frac{E}{\beta} = 0.556
\]

\[
G_c(s) = \frac{0.556}{s + 0.09} \quad s + 0.01
\]

The compensated \( \text{PM} = 40.7^\circ \)
Lag Compensator Example

Lag Compensator Example