Theorem 1: Indefinite Integration of Analytic Function:

Let \( f(z) \) be analytic in a simply connected domain \( D \). Then there exists an indefinite integral of \( f(z) \) in the domain \( D \), that is, an analytic function \( F(z) \) such that \( F'(z) = f(z) \) in \( D \), and for all paths in \( D \) joining two points \( z_0 \) and \( z_1 \) in \( D \) we have

\[
\int_C f(z)\,dz = \int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)
\]

Basically, the integration is independent of the path \( C \).
Theorem 2: Integration by Use of the Path

Let $C$ be a piecewise smooth path, represented by $z = z(t)$, where $a \leq t \leq b$. Let $f(z)$ be a continuous function on $C$. Then

$$\int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt$$

$$\dot{z}(t) = \frac{dz}{dt}$$

**Proof:** $z = x + iy$

$$\dot{z}(t) = \frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt} = \dot{x}(t) + i\dot{y}(t)$$

$$f(z) = u[x, y] + iv[x, y]$$

$$f[z(t)] = u[x(t), y(t)] + iv[x(t), y(t)]$$

$$\int_a^b f[z(t)]\dot{z}(t)dt = \int_a^b [u + iv][\dot{x} + i\dot{y}]dt$$

$$\int_c [u + iv][dx + idi] = \int_c [udx - vdy ] + i[vdx + udy]$$