Classical Mechanics

Chapter 8. The Hamilton Equations of Motion

Homework 4

Problem 4A. **(10 points)**

A spherical pendulum consists of a particle of mass $\boldsymbol{m}$ in a gravitational field constrained to move on a surface of a sphere of radius $\boldsymbol{l}$. Use the polar angle $\boldsymbol{\theta}$ (measured from the downward vertical) and the azimuthal angle $\boldsymbol{\varphi}$ to obtain the equations of motion in the Hamiltonian formulation. Expand the Hamiltonian to second order (or the Hamilton equations to the first order) about uniform circular motion with $\boldsymbol{\theta = \theta_0}$ and show that the resulting expression is just that for a simple harmonic oscillator with

$$\omega^2 = \left[{}^{g}/_{l\cos\theta_0}\right](1 + 3\cos^2\theta_0)$$

Problem 4B. **(10 points)**

A dynamical system has the Lagrangian

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1^2}\right) - \frac{1}{2}(k_1 q_1^2 + k_2)$$

Where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{k_1}, \boldsymbol{k_2}$ are constants.

a) Find a Hamiltonian corresponding to this Lagrangian.

b) What quantities are conserved?

c) Find the equations of motion in the Hamiltonian formulation and solve them.

Problem 4C. **(10 points)**

Consider the motion of a particle P of mass m moving in the plane under the influence of a force of magnitude $\alpha m/r^2$ directed towards a fixed point O, where r is the distance from O to P. Where α *is* a constant. Assume that the potential energy is zero as $r \to \infty$.

a) Find a Lagrangian.

b) Find a Hamiltonian corresponding to this Lagrangian.

c) What quantities are conserved?

d) Find the equations of motion in the Hamiltonian formulation.

e) Write down the equation for r

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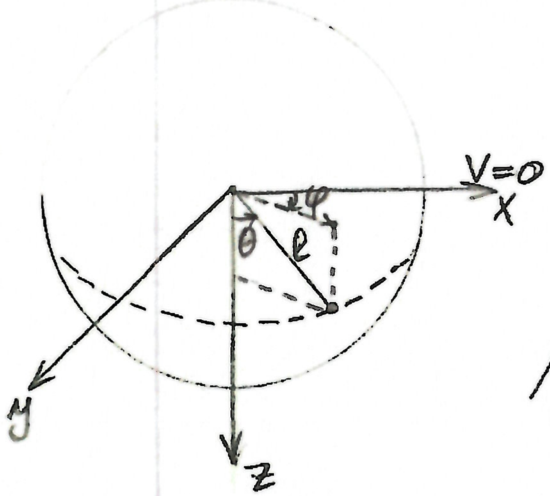
A spherical pendulum of a particle m in a grav field constrained to move on the surface of a sphere of radius l. Use θ, φ to obtain the eqs of motion in the hamiltonian formulation.



$r = l = \text{const} \qquad L = T - V$

$$L = \tfrac{1}{2} m (\dot r^2 + r^2 \dot\theta^2 + r^2 \sin^2\theta \cdot \dot\varphi^2) - (-mgl \cos\theta)$$

$V = 0$ on the plane (xy)

$$\boxed{L = \tfrac{1}{2} m (l^2 \dot\theta^2 + l^2 \sin\theta \, \dot\varphi^2) + mgl\cos\theta}$$

$$\tilde H = \sum_\alpha p_\alpha \cdot \dot q_\alpha - L$$

$$p_\alpha = \frac{\partial L}{\partial \dot q_\alpha} \Rightarrow 1)\ p_r = \frac{\partial L}{\partial \dot r} = 0 \Rightarrow p_r = 0$$

$$2)\ p_\theta = \frac{\partial L}{\partial \dot\theta} = ml^2 \dot\theta \Rightarrow \dot\theta = \frac{p_\theta}{ml^2}$$

$$3)\ p_\varphi = \frac{\partial L}{\partial \dot\varphi} = ml^2 \sin^2\theta \, \dot\varphi \Rightarrow \dot\varphi = \frac{p_\varphi}{ml^2 \sin^2\theta}$$

$$H = p_r \cdot \dot r + p_\theta \dot\theta + p_\varphi \cdot \dot\varphi - (1/2) m (l^2 \dot\theta^2 + l^2 \sin\theta \, \dot\varphi^2) - mgl\cos\theta$$

feed here $\dot r, \dot\theta, \dot\varphi$

$$H = \frac{p_\theta^2}{ml^2} + \frac{p_\varphi^2}{ml^2 \sin^2\theta} - \frac{1}{2} m \left(l^2 \cdot \frac{p_\theta^2}{m^2 l^4} + l^2 \sin^2\theta \cdot \frac{p_\varphi^2}{m^2 l^4 \sin^4\theta} \right) - mgl\cos\theta$$

$$\boxed{H = \frac{p_\theta^2}{2ml^2} + \frac{p_\varphi^2}{2ml^2 \sin^2\theta} - mgl\cos\theta} = T + V$$ (as it should be in a conservative field)

eqs. of motion are:

$$\dot q_\alpha = \frac{\partial H}{\partial p_\alpha} \Rightarrow \dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \ ; \ \dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{ml^2 \sin^2\theta} \ ; \ \begin{matrix} p_\theta = ml^2 \dot\theta \\ p_\varphi = ml^2 \sin^2\theta \, \dot\varphi \end{matrix}$$

$$\dot p_\alpha = -\frac{\partial H}{\partial q_\alpha} \Rightarrow \dot p_\theta = -\frac{\partial H}{\partial \theta} = \boxed{\frac{p_\varphi^2 \cos\theta}{ml^2 \sin^3\theta} - mgl \sin\theta = ml^2 \ddot\theta} \quad (1)$$

$$\dot p_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \Rightarrow \boxed{p_\varphi = \text{const} = ml^2 \sin^2\theta \cdot \dot\varphi} \quad (2)$$

Expand the hamiltonian to second order about uniform circular motion $\theta=\theta_0$ and show that the resulting expression is just that for a simple harmonic oscillator with $\omega^2=\left(\frac{g}{l\cos\theta_0}\right)(1+3\cos^2\theta_0)$

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so $\theta=\theta_0 \Rightarrow \dot\theta_0=0 \Rightarrow \ddot\theta_0=0$

and from the eq. (1) we can find P_φ (which is const for this problem)

$$0=\frac{P_\varphi^2\cos\theta_0}{ml^2\sin^3\theta_0}-mgl\sin\theta_0 \Rightarrow P_\varphi^2=\frac{m^2gl^3\sin^4\theta_0}{\cos\theta_0} \qquad (3)$$

now return to (1) and expand it near θ_0

$\boxed{\theta=\theta_0+\eta} \Rightarrow \theta-\theta_0=\eta$

$$\left(\frac{P_\varphi^2}{ml^2}\right)\left(\frac{\cos(\theta_0+\eta)}{\sin^3(\theta_0+\eta)}\right)-mgl\sin(\theta_0+\eta)=ml^2(\ddot\theta_0+\ddot\eta)$$

$$\sin(\theta_0+\eta)=\sin\theta_0\cdot\cos\eta+\cos\theta_0\cdot\sin\eta=\sin\theta_0\cdot 1+\cos\theta_0\cdot\eta$$

$$f(\theta)=\frac{\cos\theta}{\sin^3\theta} \Rightarrow f(\theta)=f(\theta_0)+\frac{1}{1!}f'(\theta_0)(\theta-\theta_0)+\frac{1}{2!}f''(\theta_0)(\theta-\theta_0)^2+\dots$$

$$f(\theta)=\frac{\cos\theta_0}{\sin^3\theta_0}+\left(\frac{-\sin^4\theta_0-3\sin^2\theta_0\cos^2\theta_0}{\sin^6\theta_0}\right)\eta+\left(\frac{4\cos\theta_0\sin^2\theta_0+6\cos^3\theta_0}{\sin^5\theta_0}\right)\eta^2$$

omit it

$$ml^2\ddot\eta=\frac{P_\varphi^2}{ml^2}\left(\frac{\cos\theta_0}{\sin^3\theta_0}-\frac{\sin^2\theta_0+3\cos^2\theta_0}{\sin^4\theta_0}\eta\right)-mgl\sin\theta_0-\eta mgl\cos\theta_0$$

$$ml^2\ddot\eta=\left(\frac{m^2gl^3\sin^4\theta_0}{\cos\theta_0\, ml^2}\right)\left(\frac{\sin\theta_0\cos\theta_0-\eta(\sin^2\theta_0+3\cos^2\theta_0)}{\sin^4\theta_0}\right)-mgl\sin\theta_0-\eta\cos\theta_0\, mgl$$

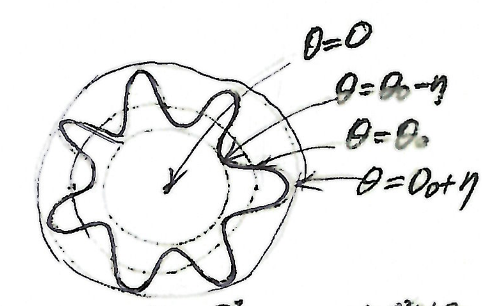
$$ml^2\ddot\eta=\cancel{mgl\sin\theta_0}-\eta g l m\frac{\sin^2\theta_0}{\cos\theta_0}-3mgl\eta\cos\theta_0-\cancel{mgl\sin\theta_0}-\eta mgl\cos\theta_0$$

$$ml^2\ddot\eta=-\eta mgl\left(\frac{\sin^2\theta_0}{\cos\theta_0}+4\cdot\cos\theta_0\right)=-\eta mgl\left(\frac{\sin^2\theta_0+4\cos^2\theta_0}{\cos\theta_0}\right)$$

$$\ddot\eta=-\eta\left(\frac{g}{l}\right)\cdot\left(\frac{1+3\cdot\cos^2\theta_0}{\cos\theta_0}\right)$$

$$\ddot\eta+\left[\left(\frac{g}{l\cdot\cos\theta_0}\right)(1+3\cos^2\theta_0)\right]\eta=0$$

$$\ddot\eta+\omega^2\eta=0$$



$$\boxed{\omega^2=\frac{g}{l\cos\theta_0}(1+3\cos^2\theta_0)}$$

from (3) and (2) we get $\dot\varphi^2=\frac{P_\varphi^2}{m^2l^4\sin^4\theta_0}=\frac{m^2gl^3\sin^4\theta_0}{m^2l^4\sin^4\theta_0\cos\theta_0}$

$\eta=A\cdot\cos(\omega t+\varphi_0)$, it oscillates near θ_0 $\quad \dot\varphi^2=\frac{g}{l\cos\theta_0} \Rightarrow \boxed{\varphi=\sqrt{\frac{g}{l\cos\theta_0}}\cdot t+C}$

φ is increasing, it's rotating.

1. **(10 points)** A dynamical system has the Lagrangian

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1^2}\right) - \frac{1}{2}(k_1 q_1^2 + k_2)$$

Where a, b, and k_1, k_2 are constants.

a) Find a Hamiltonian corresponding to this Lagrangian.
b) What quantities are conserved?
c) Find the equations of motion in the Hamiltonian formulation. and solve them.

a) Since $\mathcal{L} \neq \mathcal{L}(t)$, $V = V(q_i)$ only and T is quadratic in $\dot{q}_i$,
$\mathcal{H} = T + V$

$$\tilde{\mathcal{H}} = \frac{1}{2}\cdot\left(\dot{q}_1^2 + \frac{\dot{q}_2^2}{a + b\cdot q_1^2}\right) + \frac{1}{2}(k_1 q_1^2 + k_2)$$

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \dot{q}_1 \quad ; \quad p_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \frac{\dot{q}_2}{a + b\cdot q_1^2} \Rightarrow \dot{q}_2 = p_2\cdot(a + bq_1^2)$$

$$\mathcal{H} = \frac{p_1^2}{2} + \frac{p_2^2}{2}(a + bq_1^2) + \frac{1}{2}\cdot(k_1 q_1^2 + k_2)$$

b) Energy and p_2 momentum are conserved, since there is no explicit time dependence in $\mathcal{L}$ and q_2 is cyclic.

c) ① $\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -bq_1\cdot p_2^2 - k_1 q_1$

② $\dot{q}_1 = \frac{\partial H}{\partial p_1} = p_1 \Rightarrow \| \text{combine with ①} \| \Rightarrow \dot{p}_1 = \underline{\ddot{q}_1 = -q_1\cdot(k_1 + bp_2^2)}$

③ $\dot{p}_2 = -\frac{\partial H}{\partial q_2} = 0 \Rightarrow p_2 = \text{const} = \alpha$

④ $\dot{q}_2 = \frac{\partial H}{\partial p_2} = p_2\cdot(a + b\cdot q_1^2)$

Thus, we have 2 eq-ns :
$$\begin{cases} \dot{q}_2 = \alpha(a + bq_1^2) \\ \ddot{q}_1 = -q_1(b\cdot\alpha^2 + k_1) \end{cases}$$
where I used $p_2 = \alpha$.

- $\ddot{q}_1 = -(b\alpha^2 + k_1) q_1$

a typical "probe" for $q_1 \sim e^{kq_1}$

$$k^2 = -(b\alpha^2 + k_1) \Rightarrow k = \pm i \sqrt{b\alpha^2 + k_1}$$

so, the general solution is

$$q_1(t) = A_1 \cdot \sin \sqrt{b\alpha^2 + k_1}\, t + A_2 \cdot \cos \sqrt{b\alpha^2 + k_1}\, t$$

denote $\boxed{\omega \equiv \sqrt{b\alpha^2 + k_1}}$

$$q_1(t) = A_1 \sin \omega t + A_2 \cdot \cos \omega t$$

or it can be easily rewritten to this form

$\boxed{q_1(t) = B \cdot \sin(\omega t + \delta)}$, B, δ – can be found from the initial conditions.

- $\dot{q}_2 = \dfrac{dq_2}{dt} = \alpha(a + b q_1^2) = \alpha a + \alpha \cdot b \cdot B^2 \cdot \sin^2(\omega t + \delta)$

$$q_2(t) = \alpha \cdot a \cdot t + \alpha \cdot b \cdot B^2 \int \left(\frac{1 - \cos 2(\omega t + \delta)}{2} \right) dt =$$

$$= \alpha a t + \alpha b B^2 \frac{t}{2} - \frac{\alpha b B^2}{2} \cdot \frac{1}{2\omega} \cdot \sin 2(\omega t + \delta) + C_1$$

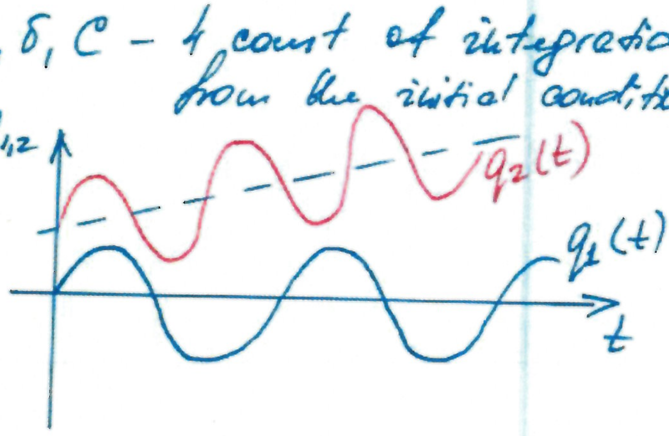
const of integration

$$q_2(t) = \alpha\left(a + \frac{bB^2}{2}\right) t - \frac{\alpha b B^2}{4\omega} \sin 2(\omega t + \delta) + C$$

$$q_1(t) = B \cdot \sin(\omega t + \delta)$$

$\omega = \sqrt{b\alpha^2 + k_1}$; $\alpha = p_2 = \text{const}$

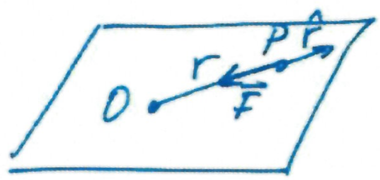
α, B, δ, C – 4 const of integration which can be found from the initial conditions.



1. Consider the motion of a particle P of mass m moving in the plane under the influence of a force of magnitude $\alpha m/r^2$ directed towards a fixed point O, where r is the distance from O to P. Where α *is* a constant.

a) Find a Lagrangian.
b) Find a Hamiltonian corresponding to this Lagrangian.
c) What quantities are conserved?
d) Find the equations of motion in the Hamiltonian formulation.
e) Write down the equation for *r*

a) $\vec{F} = -\frac{\alpha m}{r^2}\hat{r}$



$V = -\int_{r_0}^{r} \vec{F}\cdot d\vec{r} = \int_{r_0}^{r} \frac{\alpha m}{r^2} dr = -\frac{\alpha m}{r}\Big|_{r_0}^{r} =$

$= -\alpha m\left(\frac{1}{r} - \frac{1}{r_0}\right) = \left\| \begin{matrix}\text{Since}\\ V=0 \text{ as}\\ r_0 \to \infty\end{matrix} \right\| = -\frac{\alpha m}{r}$, so

$$\boxed{\mathcal{L} = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\alpha m}{r}}$$

r, θ – polar coordinates with origin O.

b) $p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$ $\quad p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$

$\dot{r} = p_r/m$; $\dot{\theta} = p_\theta/mr^2$

$\tilde{\mathcal{H}} = p_r\dot{r} + p_\theta\cdot\dot{\theta} - \mathcal{L} = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{m}{2}\cdot\left(\frac{p_r^2}{m^2} + r^2\cdot\frac{p_\theta^2}{m^2r^4}\right) - \frac{\alpha m}{r} =$

$$\boxed{\mathcal{H} = \frac{p_r^2}{2\cdot m} + \frac{p_\theta^2}{2mr^2} - \frac{\alpha m}{r}}$$

• Hamilton's eq-ns

① $\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} \Rightarrow \dot{r} = p_r/m$

② $\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} \Rightarrow \dot{\theta} = p_\theta/mr^2$

③ $\dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\theta^2}{m \cdot r^3} - \frac{\alpha m}{r^2} =$

④ $\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = 0 \Rightarrow p_\theta = \text{const} = l$

from ① $\Rightarrow \dot{p}_r = m\ddot{r}$, put it in ③

$$\boxed{m\ddot{r} = \frac{l^2}{mr^3} - \frac{\alpha m}{r^2}}$$

Energy is conserved.
Angular momentum (p_θ) is conserved.