Harmonic motion is a type of motion in which an object moves along a repeating path over and over again.

The universe is filled with examples of harmonic motion:
- Pendulum motions
- The motion of the Earth as it orbits the Sun

Certain features involving force and energy are common to all forms of harmonic motion.
General Features

- The spring is the dominant force on the person
  - Dominates over gravity
- The person vibrates up and down due to the force of the spring
- The position of the person can be described by measuring his location $y$ on a vertical axis

![Diagram showing oscillatory motion of a spring-mass system with positions marked as $y = A$, $y = -A$, Highest point, Lowest point.](image)
General Features, cont.

- The plot shows position as a function of time.
- The person’s position varies in a repeating fashion as he moves up and down.
- This is an example of **oscillatory motion**.
General Features, Final

- Position, velocity, and acceleration can again be used to describe oscillatory motion.
- Other quantities are also needed:
  - **Period**, $T$
    - The repeat time of the motion.
    - The time it takes to go through one complete cycle or oscillation.
  - **Frequency**, $f$
    - The number of oscillations in one unit of time.
    - Generally use a time of one second, gives units of cycles / sec or Hertz (Hz).
  - Period and frequency are related by $f = \frac{1}{T}$.
Simple Harmonic Motion

- Systems that oscillate in a sinusoidal matter are called simple harmonic oscillators
  - They exhibit *simple harmonic motion*
  - Abbreviated SHM
- The position can be described by $y = A \sin (2\pi ft)$
  - $A$ is the amplitude of the motion
  - The object moves back and forth between the positions $y = \pm A$
  - $f$ is the frequency of the motion
Simple Harmonic Motion, Velocity

- The velocity also “oscillates” between positive and negative values.
- The frequency is the same as for the position.
- The largest value of $v$ occurs at $y = 0$.
- At $y = \pm A$, $v = 0$.
SHM and Circular Motion

- A spinning DVD is another example of periodic, repeating motion
- Observe a particle on the edge of the DVD
- The particle moves with a constant speed $v_c$
- Its position is given by $\theta = \omega t$
  - $\omega$ is the angular velocity

Section 11.1
SHM and Circular Motion, cont.

- As the particle completes one full trip, it will travel $2\pi$ radians and take one period

$$\theta = 2\pi = \omega T$$

$$T = \frac{2\pi}{\omega} \text{ and } f = \frac{\omega}{2\pi}$$

- $\omega$ is also called the angular frequency
SHM: Position as a Function of Time

- Chart the motion of the particle on the DVD in the x-y plane.
- The y-component is
  \[ y = A \sin \theta = A \sin (\omega t) = A \sin (2\pi f t) \]
- The x-component is
  \[ x = A \cos \theta = A \cos (\omega t) = A \cos (2\pi f t) \]
- The only difference is the way in which the system is initially set into motion.

Both curves describe simple harmonic motion.

Section 11.1
SHM: Velocity as a Function of Time

- Look at the y-component of the particle’s velocity
- Although the speed of the particle is constant, its y-component is not constant
  \[ v_y = v_c \cos \theta = v_c \cos(\omega t) = v_c \cos (2 \pi f t) \]
- From the definition of velocity, we can express the particle’s speed as
  \[ v = 2 \pi f A \cos (2 \pi f t) \]
SHM: Equation Summary

- The position and velocity of a simple harmonic oscillator can be described by two sets of equations:
  - \( y = A \sin (2 \pi f t) \) and \( v = 2 \pi f A \cos (2 \pi f t) \)
  - \( x = A \cos (2 \pi f t) \) and \( v = -2 \pi f A \sin (2 \pi f t) \)
- These relationships both describe simple harmonic motion.
- The difference between them is how the oscillator is initially set into motion.

Section 11.1
SHM Example: Mass on a Spring

- Assume a block is moving on a frictionless surface.
- The force exerted by the spring is given by Hooke's Law: \( F_{\text{spring}} = -kx \)
  - \( x \) is the amount the spring is stretched or compressed away from its relaxed length.
The block is in equilibrium at $x = 0$.

If the block is displaced, the spring force will act in the opposite direction.

This force is called a *restoring force* because it always opposes the displacement away from the equilibrium position.
Mass on a Spring, final

- Whenever there is a restoring force that obeys Hooke’s Law, the system will exhibit simple harmonic motion.
- Applying Newton’s Second Law: \( F = -kx = ma \)
- From the comparison to circular motion, insert the expression for the velocity, the frequency can be found:

\[
a = -a_c \cos \theta = -\frac{v_c^2}{A} \cos \theta
\]

- Inserting the expression for the velocity, the frequency can be found:

\[
f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \text{and} \quad T = \frac{1}{f}
\]

- Note the frequency is independent of the amplitude.
Mass on a Vertical Spring

- When the spring is hung vertically, gravity and the spring force act on the object.
- The oscillation frequency for a mass on a vertical spring is precisely the same as the frequency for a mass on a horizontal spring.
- The frequency is still independent of the amplitude of the motion.
SHM Example: Simple Pendulum

- With an object whose diameter is small compared to the length of the string and the string is massless, an object tied to a string makes a system that can be considered a simple pendulum.
- Gravity and the tension in the string act on the object.
- The total force is parallel to the arc and is responsible for the simple harmonic motion.
- The bottom point of the trajectory is the equilibrium position.
The total force is always directed toward the bottom of the arc. That is the *equilibrium point* of the pendulum.
Simple Pendulum: Analysis

- The restoring force is the component of gravity directed along the body’s path:
  \[ F_{\text{restore}} = F_{\text{parallel}} = -mg \sin \theta \]
- If the angle is small, \( \sin \theta \approx \theta \)
  - When the angle is measured in radians
- The angle is also related to the displacement by \( \theta = \frac{y}{L} \)
- Applying Newton’s Second Law:
  \[ F_{\text{parallel}} = -\frac{mgy}{L} = ma \]
  - The force is proportional to the displacement
Simple Pendulum: Frequency

- The frequency of a simple pendulum is given by
  \[ f = \frac{1}{2\pi} \sqrt{\frac{g}{L}} \]
- The frequency and period are independent of the mass of the pendulum bob
- The frequency and period are independent of the amplitude of the motion
- However, small angles were assumed
  - As long as the angle is under \( \sim 30^\circ \), this equation is a good approximation
A torsional oscillator also displays simple harmonic motion.

The wire shown in the examples is called a torsion fiber.

When it is twisted, it exerts a torque on whatever it is connected to.
Torsional Oscillator, Analysis

• The torque is proportional to the twist angle: $\tau = -\kappa \theta$
  - $\kappa$ is called the torsion constant and is a property of the specific torsion fiber
  - The negative sign indicates a restoring torque
  - This equation has the same form as Hooke’s Law

• The frequency of a torsional oscillator is given by

$$f = \frac{1}{2\pi} \sqrt{\frac{\kappa}{I}}$$

  - $I$ is the moment of inertia of the object attached to the fiber
Features Common to All Simple Harmonic Oscillators

- Their motion exhibits a sinusoidal time dependence
- They all involve a restoring force
  - F always has a sign opposite the displacement
- Whenever an object experiences a restoring force with a magnitude proportional to the displacement from the equilibrium position, the object will undergo simple harmonic motion
- The frequency of all simple harmonic oscillators is independent of the amplitude of the motion
  - The amplitude is determined by how the oscillator is initially set into motion
Harmonic Motion and Energy

<table>
<thead>
<tr>
<th>KE</th>
<th>PE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}mv_{\text{max}}^2$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{1}{2}kA^2$</td>
</tr>
<tr>
<td>$\frac{1}{2}mv_{\text{max}}^2$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{1}{2}kA^2$</td>
</tr>
</tbody>
</table>

$\nu = \nu_{\text{max}}$

$\nu = 0$

$\nu = -\nu_{\text{max}}$

$\nu = 0$

$x = 0$

$x = +A$

$x = 0$

$x = -A$
Energy, cont.

• The mechanical energy is the sum of the kinetic and potential energies
• When $x = 0$, the potential energy is 0 and the kinetic energy is a maximum
• When the displacement is $\pm A$, the kinetic energy is 0 and the potential energy is a maximum
• For an ideal oscillator, the total mechanical energy is conserved
  • No friction is present
• The maximum kinetic energy must also equal the maximum potential energy:
\[ \frac{1}{2} m v_{\text{max}}^2 = KE_{\text{max}} = PE_{\text{max}} = \frac{1}{2} k A^2 \]
• These results apply to all simple harmonic oscillators
Stress, Strain, and Hooke’s Law

- Hooke’s Law for a spring is \( F = -kx \)
- This applies to many real situations as well as to ideal springs
  - For example, if a metal bar is squeezed, it “pushes back” with a force that depends on how much you squeezed it
  - The force is given by Hooke’s Law
Stress and Strain, cont.

- When the force is applied to the bar, the bar changes length by an amount $\Delta L$
  - The force shown is a compressive force, and $\Delta L$ is negative
  - A stretching force would be a tensile force and $\Delta L$ would be positive

Section 11.4
The value of $\Delta L$ depends on the material.

The “stiffness” of the material is characterized by Young’s Modulus, $Y$.

For a wide variety of materials, the relationship between the magnitude of the force and the change in length is given by

$$\frac{F}{A} = Y \frac{\Delta L}{L_o}$$

The equation applies to both compressive and tensile forces.
Stress and Strain Defined

• The ratio $F / A$ is called the stress
  • It allows for the easier comparison of bars of different sizes

• The ratio $\Delta L / L_0$ is called the strain
  • It is the fractional change in the length
  • This will be a unitless ratio

• The value of Young’s Modulus depends on the material
  • See table 11.1 for values for common materials
  • Units for $Y$ are $N/m^2 = Pa$
    • These are the units for the stress, too

Section 11.4
Elastic Deformations

- Y is called an elastic modulus
- The material will return to its original length when the force is removed
- This is the behavior of most materials if the strain is not too large

When the elastic limit is exceeded, the stress–strain curve is no longer linear.

Linear relation $\Rightarrow$ Hooke’s law

Section 11.4
Plastic Deformations

• If the strain exceeds a certain value, the linear relationship between stress and strain in Hooke’s Law is no longer valid
• The strain at which this happens is called the *elastic limit*
  • The value depends on the material
• If the elastic limit is exceeded, the material will be permanently deformed
Shear Modulus

- A shearing force may be applied to a bar.
- This will tend to make two parallel faces shift laterally by an amount $\Delta x$.
- Hooke’s Law becomes:
  \[ \frac{F}{A} = S \frac{\Delta x}{L_0} \]
- $S$ is the shear modulus.
  - See table 11.1 for values.

\[ A = \text{cross-sectional area} \]

This end is held fixed.

Section 11.4
Bulk Modulus

- An object may be subject to a force that tends to change its overall size.
- The force is normal to the object’s surface and causes a change in its volume of $\Delta V$.

$$P = -B \frac{\Delta V}{V_o}$$

$V_0 = \text{original volume}$
Bulk Modulus, cont.

• The pressure and change in volume are directly proportional

• The negative sign indicates that a positive pressure will produce a negative change in volume
  • Increasing the pressure decreases the volume

• A bulk modulus applies to solids, liquids, and gases
  • All phases change volume in response to changes in pressure
  • Compressive, tensile and shearing stresses can only be applied to objects that maintain their shapes without the help of a container

Section 11.4
Elastic Properties and SHM

• If the end of a bar is displaced it experiences a restoring force proportional to $\Delta L$
  • For a compressive or tensile force, the restoring force involves Young’s Modulus
  • For a shear displacement, the restoring force involves the Shear Modulus
  • Some distortions may involve both
• In all these cases, the restoring force is described by Hooke’s Law
• A vibrating metal bar is closely analogous to a mass on a spring and generally behaves like a simple harmonic oscillator

Section 11.4
The friction in an oscillating system is referred to as **damping**.
The motion is referred to as a **damped harmonic oscillator**.
Depending on the amount of damping, different motions are observed as seen in the graph.
Underdamped System

- When there is only a small amount of friction, the damping is weak
- The system will oscillate back and forth, but its amplitude will gradually decrease
- The amplitude eventually goes to zero
- This motion is called underdamped oscillation
- See curve 3 in figure 11.26
Overdamped System

- When there is a large amount of friction, the damping is strong.
- When released, the bob swings very slowly to the bottom.
- The object does not oscillate.
- This motion is called *overdamped* oscillation.
- See curve 1 in figure 11.26.
Damping, Summary

- **Underdamped**
  - The displacement always passes through the equilibrium position before the system comes to rest
  - Generally many times

- **Overdamped**
  - The system does not pass through the equilibrium position
  - It moves slowly to the equilibrium position without going past it

- **Critically damped**
  - The boundary between the other two cases
  - The displacement falls to zero as rapidly as possible without moving past the equilibrium position
  - See curve 2 in figure 11.25

Section 11.5
Damping, Shock Absorbers Example

- Shock absorbers are springs that support a car’s weight
- They allow the tires to move up and down according to the bumps in the road without passing the vibrations in to the car and its passengers
- They are designed to be critically damped
  - Allows the body of the car to return to its original height as quickly as possible
- If underdamped
  - The car oscillates up and down
Driven Oscillator

- An oscillator can be subjected to a force applied at regular intervals
  - In this case, the oscillator is said to be *driven*
- Assume the driving force is applied with a frequency of \( f_{\text{drive}} \)
- If the frequency of the driving force is not close to the natural frequency of the system, the amplitude is small
- The amplitude is largest when the frequency of the driving force matches the natural frequency of the system
Driven Oscillators, cont.

- **Resonance** occurs when the frequencies match.
- A resonance curve shows the peak occurs at the **resonant frequency**.
- The heights and widths of the peaks depend on the damping.

![Resonance Curve](image)

- Weak damping
- Medium damping
- Strong damping

Amplitude vs. Driving frequency

Section 11.5
Detecting Small Forces

- Oscillations and elasticity come together in the problem of detecting small forces
- Two examples
  - The Cavendish experiment to measure the force of gravity
  - The atomic force microscope
The Cavendish Experiment

- Two masses (metal spheres) were mounted on a light rod
- The rod was suspended by a thin wire
- The force of gravity exerted by other spheres brought close to the mounted ones caused the torsion fiber to twist
- The twist angle and $F_{\text{grav}}$ could be measured
Atomic Force Microscope

- A small bar is fashioned so that it has a very small tip at the end
  - The bar is called a cantilever
- It bends in response to any force that acts on it
  - The forces usually act at the tip, so the cantilever bends up and down
- By measuring the cantilever’s deflections as a function of its position, an image of the surface can be developed

Section 11.6
The cantilever is scanned over the surface.

The AFM tip is deflected up and down as it passes over atoms.