1. Overview of Electrodynastic Theories

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2. Class Overview

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- Static charges create static electric fields.
- Steadily moving charges create static magnetic fields.
- Accelerating charges create changing fields.
- Relativistic corrections to accelerating charges creating changing fields.
- Charged particles in quantum states interact with quantized electromagnetic field.
- Relativistic corrections to charged particles in quantum states interacting with the quantized electromagnetic field.

3. Macroscopic Theory

- Quantum theory is required to accurately describe atomic-scale electrodynamics.
- Classical electrodynamics is therefore a macroscopic theory.
- There is no valid concept of an electron in classical electrodynamics: the word “electron” is hereby banned from this class. Instead we speak of large collections of electrons that behave macroscopically known as “electric charge”.
- For mathematical consistency, point charges are defined, but they do not represent electrons.
- The point charge $q$ represents a macroscopic charged object that is small enough that the
charge can be approximated to be found at its exact center, but large enough that it behaves classically.
- The line charge density $\lambda$ is similarly a mathematical construct where all the charge along a wire is considered to be at the wire's exact axis.
- The surface charge density $\sigma$ is lastly a mathematical construct where all the charge in the surface is considered to be in the exact center of the surface's thickness.

4. Dirac Delta Distributions

- Definition: $\delta(x-x_0)=0$ for $x \neq x_0$ and $\int \delta(x-x_0) f(x) \, dx = f(x_0)$
- Multiple Dimensions: $\delta^{(2)}(x)=\delta(x-x_0)\delta(y-y_0)$, $\delta^{(3)}(x)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$
- The dimensionality superscript is often omitted when it is clear from the context.
- Point charge: $\rho(x)=q_i \delta^{(3)}(x-x_i)$ where $x_i$ is the position of the charge.
- Line charge: $\rho(x)=\lambda(x)\delta^{(2)}$
- Surface charge: $\rho(x)=\sigma(x)\delta^{(1)}$
- Dirac deltas obey the property: $\delta(au) = \frac{1}{|a|} \delta(u)$
- This shows us that the units of the Dirac delta are the inverse of the units of its operand. We could have guessed as much by examining its integral definition. It is very useful to check an equation as you solve a problem by making sure the units work out, and this identity facilitate a units check.
- The general expression for the Dirac delta in any orthogonal coordinate system $(u, v, w)$ with length elements $(du/U, dv/V, dw/W)$ is given by:
  $$\delta(x-x') = \delta(u-u')\delta(v-v')\delta(w-w')U V W$$
- For example, in spherical coordinates $(r, \theta, \phi)$ the length elements are $(dr, r \, d\theta, r \sin \theta \, d\phi)$ so:
  $$\delta(x-x') = \delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') \frac{1}{r^2 \sin \theta}$$
- If we needed only a two dimensional delta in spherical coordinates (for example, to represent a line charge shaped into a circle), we would omit the delta and length element in the direction where the charge distribution extends.
5. **Coulomb's Law**

- Coulomb showed experimentally that for two electrical point charges exerting a force on each other:
  - force ∝ charge
  - force ∝ 1/(distance$^2$)
  - force is central
  - force is attractive for oppositely charge bodies

- Create the equation: $F \propto q_1 q_2 \rightarrow F \propto \frac{q_1 q_2}{r^2} \rightarrow F \propto \frac{q_1 q_2}{r^2} \hat{r} \rightarrow F = k \frac{q_1 q_2}{r^2} \hat{r}$

- Convert this to a fixed coordinate system to be mathematically useful:

$$F = k \frac{q_1 q_2}{|x_1 - x_2|} \hat{r}$$

$$F = k \frac{q_1 q_2 (x_1 - x_2)}{|x_1 - x_2|^2 |x_1 - x_2|}$$

$$F = k \frac{q_1 q_2 (x_1 - x_2)}{|x_1 - x_2|^3}$$

- Be careful not to let the 3 in the exponent of the denominator deceive you. This is still the inverse square law because one of the powers will cancel with the magnitude of the numerator.

- The proportionality constant $k$ is dependent on the units used and is found experimentally.

- For SI units, $k = \frac{1}{4 \pi \varepsilon_0}$ in vacuum.

- The final form becomes:

$$F = \frac{1}{4 \pi \varepsilon_0} \frac{q_1 q_2 (x_1 - x_2)}{|x_1 - x_2|^3}$$

**Coulomb's Law, in terms of the force, for two charges**

- To be more useful, factor out one charge and define the electric field $E$:

$$F = q_1 \left[ \frac{1}{4 \pi \varepsilon_0} \frac{q_2 (x_1 - x_2)}{|x_1 - x_2|^3} \right]$$

$$F = q_1 E$$

where

$$E(x) = \frac{1}{4 \pi \varepsilon_0} \frac{q (x - x_1)}{|x - x_1|^3}$$

**Coulomb's Law, in terms of the field, for one charge**
The electric field is a vector field. It is defined at every point in space and can therefore be expressed as a function of spatial coordinates. Every point in space has an electric field vector with direction and magnitude. But the force is not a field of vectors, but a single vector. It is the force felt at the location of the charge \( q_1 \). What if we want to know the force exerted on an extended charge distribution \( \rho(x) \) when placed in an electric field \( E \)? Then we must sum over all the little infinitesimal bits of charge that make up the extended object:

\[
F = \int \rho(x) E(x) d^3x
\]

- When using this equation, there is still just one net force vector \( F \) that the object with charge distribution \( \rho(x) \) experiences.
- Now that we know how to handle forces exerted by fields, we can focus only on the electric field as the entity to be solved for.
- Experimental observation shows that fields add linearly, therefore the total electric field is the vector sum:

\[
E(x) = \frac{1}{4 \pi \epsilon_0} \sum_i q_i \frac{(x-x_i)}{|x-x_i|^3}
\]

*Coulomb's Law, in terms of the field, for many charges*

- In the mathematical limit of increasingly smaller charges, the sum becomes an integral

\[
E(x) = \frac{1}{4 \pi \epsilon_0} \int \frac{(x-x')}{|x-x'|^3} dq
\]

\[
E(x) = \frac{1}{4 \pi \epsilon_0} \int \frac{(x-x')}{|x-x'|^3} d^3x' d^3x
\]

\[
E(x) = \frac{1}{4 \pi \epsilon_0} \int \frac{(x-x')}{|x-x'|^3} \rho(x') d^3x
\]

*Coulomb's Law, in terms of the field, for a charge density*

- Now use different forms of the charge density:

- Using \( \rho(x')=\sigma(x') \delta^{(1)} \) gives

\[
E(x) = \frac{1}{4 \pi \epsilon_0} \int \frac{(x-x')}{|x-x'|^3} \sigma(x') d^2x'
\]

- Using \( \rho(x')=\lambda(x') \delta^{(2)} \) gives

\[
E(x) = \frac{1}{4 \pi \epsilon_0} \int \frac{(x-x')}{|x-x'|^3} \lambda(x') d^4x
\]

- Using \( \rho(x')=q \delta^{(3)}(x'-x_1) \) returns the point charge equation

\[
E(x) = \frac{1}{4 \pi \epsilon_0} \frac{(x-x_1)}{|x-x_1|^3} q
\]

- Coulomb's Law involves no bounding surfaces or boundary conditions. Therefore the integral or sum over all charges must be done over the entire universe, or we must assume that all other charges in the universe are effectively infinitely far away where their influence is negligible. Because of this, Coulomb's law has limited applicability. We can cast it into forms that are more useful.
6. Gauss's Law
- Take a point charge and fix a closed surface around it.
- At some point \( x \) on the surface, there is a vector \( n \) normal to the surface.

- Take Coulomb's law for a point charge:
  \[
  E(x) = \frac{1}{4\pi \varepsilon_0} \frac{q (x-x_i)}{|x-x_i|^3}
  \]

- Put the charge at the origin:
  \[
  E = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \hat{r}
  \]

- Dot both sides by the normal vector:
  \[
  E \cdot n = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \cos \theta
  \]

- Do a surface integral over the closed surface:
  \[
  \oint E \cdot n \, da = \oint \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \cos \theta \, da
  \]

- Expand the surface element using \( \cos \theta \, da = r^2 \, d\Omega \):
  \[
  \oint E \cdot n \, da = \oint \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} r^2 \, d\Omega
  \]
  \[
  \oint E \cdot n \, da = \frac{1}{4\pi \varepsilon_0} q \oint \frac{1}{r^2} \, d\Omega
  \]
  \[
  \oint E \cdot n \, da = \frac{q}{\varepsilon_0}
  \]

  \textbf{Gauss's Law in integral form for a point charge}

- Expand into a charge distribution:
  \[
  \oint E \cdot n \, da = \frac{1}{\varepsilon_0} \int dq
  \]
  \[
  \oint E \cdot n \, da = \frac{1}{\varepsilon_0} \int \frac{dq}{d^3 x} d^3 x
  \]
\[ \oint \mathbf{E} \cdot \mathbf{n} \, da = \frac{1}{\varepsilon_0} \int \rho(x) \, d^3 x \]

Gauss's Law in integral form for a charge distribution

- The integral form of Gauss's Law allows us to know the electric field anywhere on a surface surrounding a finite charge distribution. In practice, such surface integrals are only tractable analytically for especially simple geometries.
- Divergence theorem: for any vector field \( \mathbf{A} \):
  \[ \oint \mathbf{A} \cdot \mathbf{n} \, da = \int \nabla \cdot \mathbf{A} \, d^3 x \]

- Apply the divergence theorem to the left side of Gauss's law in integral form:
  \[ \int \nabla \cdot \mathbf{E} \, d^3 x = \frac{1}{\varepsilon_0} \int \rho(x) \, d^3 x \]

- Because this must be true for any arbitrary volume over which the integral is done, we can shrink the volume down repeatedly to every point in space. Thus the integrands must be equal.

\[ \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho(x) \]

Gauss's Law in differential form

- As opposed to Coulomb's law and Gauss's law in integral form, Gauss's law in differential form is a local equation. It connects the charge density at one point and the electric field at the same point. For this reason, Gauss's law in differential form is often the most useful in practice.
- Note that if an integral is zero, that does not mean that the integrand must be zero. Think of a one-dimensional function that is sometimes positive and sometimes negative. The area under the positive part of the curve could cancel the area under the negative part of the curve and give a total integral of zero even though the function itself is usually not zero.
- For Gauss's Law in integral form, if there is no charge present that just means that the total number of field lines entering the Gaussian surface must equal the total number leaving it.

7. The Scalar Potential
- Gauss's law involves the electric field, which is a vector property. Because solving vector equations is much more difficult and involved than solving scalar equations, the mathematics can be simplified by transforming Gauss's law into a scalar form.
- Electrostatic fields are experimentally observed to be irrotational (have no curl):

\[ \nabla \times \mathbf{E} = 0 \]

Electrostatic fields are irrotational
- Interestingly, mathematics tells us that the gradient of any scalar field also has no curl:
\[ \nabla \times (\nabla \Phi) = 0 \]

- Comparing the two equations above, we can define the electric field in terms of some electrostatic scalar potential:
\[ \mathbf{E} = -\nabla \Phi \]

- It should be noted that in a strict sense, the scalar potential itself is just a mathematical entity which simplifies calculations, and has no formal physical meaning. However, differences of the scalar potential (also known as potential differences or voltages) can be related to things with physical meaning.
- The work done in moving a charge from point \( A \) to point \( B \) in the presence of an electric field is just the integrated force times distance:
\[ W = -\int_{A}^{B} \mathbf{F} \cdot d\mathbf{l} \]

- Substitute in the force \( \mathbf{F} \) in terms of the electric field, and the electric field in terms of the potential:
\[ W = -\int_{A}^{B} (q_1 \mathbf{E}) \cdot d\mathbf{l} \quad \rightarrow \quad W = q_1 \int_{A}^{B} \nabla \Phi \cdot d\mathbf{l} \quad \rightarrow \quad W = q_1 [\Phi(x_B) - \Phi(x_A)] \]

- Therefore, the potential difference between two points is the work done \( W \) to move a test charge \( q_1 \) between these two points: \( \Delta \Phi = W / q_1 \).
- By the work-energy theorem, the potential difference can also be interpreted as the potential energy \( U \) of a test charge \( q_1 \) with reference to the initial point:
\[ \Delta \Phi = U / q_1 \]

- Next note that the electrostatic scalar potential is always continuous (except across line charges and dipole charge layers – but these are unphysical idealizations).
- The continuous nature of the electric potential can be seen from the fact that a discontinuous potential would lead to an infinite slope, and therefore an infinite electric field. An infinite force would be required to push a test charge passed the discontinuity.
- The continuity of the electric potential can be used as a boundary condition on the potential but not on the fields.
- We can now transform the electrostatic equations into forms that are in terms of the scalar potential.
- Take Coulomb's Law and use the scalar potential definition:
\[ \mathbf{E}(x) = \frac{1}{4\pi \epsilon_0} \int \frac{(x - x')}{|x - x'|^3} \rho(x') d\mathbf{x}' \]
\[-\nabla \Phi = \frac{1}{4 \pi \epsilon_0} \int \frac{(x-x') \rho(x')}{|x-x'|^3} \, dx'\]

- Now use the identity: \[\frac{x-x'}{|x-x'|^3} = -\nabla \left( \frac{1}{|x-x'|} \right)\] (Prove this for yourself and save for later.)

\[-\nabla \Phi = \frac{1}{4 \pi \epsilon_0} \int \left[-\nabla \frac{1}{|x-x'|}\right] \rho(x') \, dx'\]

\[\Phi = \frac{1}{4 \pi \epsilon_0} \int \frac{\rho(x')}{|x-x'|} \, dx' + C\]

Coulomb's law in terms of the scalar potential

- The arbitrary constant is usually made to go away by defining a ground and specifying the ground as a potential of zero.
- Take Gauss's law in differential form and use the scalar potential definition:

\[\nabla \cdot (-\nabla \Phi) = \frac{1}{\epsilon_0} \rho(x)\]

Poisson Equation

- A special case of the Poisson equation results when a region has no charges:

\[\nabla^2 \Phi = 0\]

The Laplace Equation

- The potential in the Laplace Equation is defined uniquely by the boundary conditions alone.

8. Capacitance

- A special class of systems in electrostatics involves a collection of separate objects that are perfect conductors residing in free space.
- Because such systems have only conductors and free space, the only properties involved are the charges, the conductor potentials and geometric properties linking the two.
- The electrostatic potential at any point in space due to a collection of \(n\) separate objects is just the sum over the potentials due to each object:

\[\Phi = \frac{1}{4 \pi \epsilon_0} \sum_{j=1}^{n} \int \frac{\rho_j(x')}{|x-x'|} \, dx'\]

- This expression simplifies if we only care about points in space on the surface of one of the objects. The surface of each conductor is a constant value (an equipotential) \(V_i\):

\[V_i = \frac{1}{4 \pi \epsilon_0} \sum_{j=1}^{n} \int \frac{\rho_j(x')}{|x_i-x'|} \, dx'\] where \(x_i\) is a point on the surface of object \(i\).

- When dealing with conductors, the charge density tends to spread out into the same minimal-
energy pattern every time no matter how much charge is applied. For this reason, the total charge on the $j^{th}$ object $Q_j$ is an externally applied property, but the normalized charged density $\rho_j/Q_j$ is a function of the system's geometry. We factor this out explicitly:

$$V_i = \sum_{j=1}^{n \text{ objects}} \left[ \frac{1}{4 \pi \epsilon_0} \int \frac{\rho_j(x')/Q_j}{|x_i-x'|} d\mathbf{x}' \right] Q_j$$

- At this point, the quantity in brackets is purely a function of system geometry, which does not change. We can calculate it once for a certain system by using a dummy set of charges and potentials, and then know this number forevermore.

$$V_i = \sum_{j=1}^{n \text{ objects}} p_{ij} Q_j \quad \text{where} \quad p_{ij} = \frac{1}{4 \pi \epsilon_0} \int \frac{\rho_j(x')/Q_j}{|x_i-x'|} d\mathbf{x}'$$

- The variable $p_{ij}$ is the normalized electric potential felt at the $i^{th}$ conductor due to the $j^{th}$ conductor.
- This tells us that once we know the total charge we applied to each conductor, we can find their potentials.
- The opposite case is more useful in practice: we apply certain potentials to each conductor, and we want to know the total charges.
- Fortunately, the linear system of $n$ equations can be inverted to yield:

$$Q_i = \sum_{j=1}^{n \text{ objects}} C_{ij} V_j$$

- The variable $C_{ij}$ is the capacitance of the $i^{th}$ object, and the variables $C_{ij}, i \neq j$ are the coefficients of inductance.
- Because we are dealing with perfect conductors, the capacitances and coefficients of inductance do not depend on material properties, but only on the geometry (shape and relative location) of the objects.
- For example, suppose we have only two conductors (two parallel plates or two concentric cylinders) and we want to calculate the capacitance so that we can stamp it on the side of the system for future use. The general equations are:

$$Q_1 = C_{11} V_1 + C_{12} V_2$$
$$Q_2 = C_{21} V_1 + C_{22} V_2$$

- Let us focus on the first equation. We are always free to add or subtract an overall constant to all the potentials without changing the physics. Let us subtract $V_2$ from each potential.

$$Q_1 = C_{11} (V_1 - V_2)$$
$$C_{11} = \frac{Q_1}{(V_1 - V_2)}$$

- For this simple system, there is a symmetry so that if both objects have equal and opposite charge, they will have the same capacitance, so that we can write:
\[ C = \frac{Q}{\Delta V} \]

- Because \( Q \) is the total charge on one conductor and \( \Delta V \) is the potential difference between the two conductors, we can interpret the capacitance as a \textit{geometry-dependent quantity that describes the system's ability to hold charge}.
- A system with a higher capacitance can hold more charge for a given potential difference.
- When holding as much charge as possible is the aim of a capacitor, its geometry is altered to maximize the capacitance.