92.530 Applied Mathematics I: Solutions to Homework Problems in Chapter 12

- **32:** With $v(x,y) = xF(2x+y)$, we have

$$v_x = F(2x+y) + 2xF'(2x+y),$$

and

$$v_y = xF'(2x+y).$$

Then

$$xv_x - 2xv_y = xF(2x+y) + 2x^2F'(2x+y) - 2x^2F'(2x+y) = xF(2x+y) = v.$$

If $v(1,y) = y^2$, then

$$y^2 = v(1,y) = F(2+y),$$

or, substituting $t = y+2$,

$$F(t) = (t-2)^2.$$

Therefore,

$$v(x,y) = xF(2x+y) = x(2x+y-2)^2.$$

- **34(a):** We have

$$z_x = e^x(-3f'(2y-3x)) + e^x f(2y-3x) = -3e^x f'(2y-3x) + z,$$

and

$$z_y = e^x(2f'(2y-3x)) = 2e^x f'(2y-3x).$$

Therefore,

$$2z_x + 3z_y = 2z.$$

- **35:** From the equation for the string vibrating in the absence of gravity, we had

$$y_{tt} = a^2 y_{xx},$$

so the vertical force on the string at (x,t) is

$$my_{tt} = ma^2 y_{xx}.$$

When gravity is included, the additional vertical force is $-mg$. Therefore, the total vertical force becomes

$$my_{tt} = ma^2 y_{xx} - mg,$$

so that

$$y_{tt} = a^2 y_{xx} - g.$$

- **41(a):** We can write

$$xz_{xy} + z_y = \frac{\partial}{\partial y}(xz_x + z) = 0,$$

so

$$xz_x + z = f(x),$$

for some function of x only. But we also have

$$xz_x + z = \frac{\partial}{\partial x}(xz),$$

so

$$\frac{\partial}{\partial x}(xz) = f(x),$$

so that

$$xz = F(x) + G(y),$$

for $F'(x) = f(x)$.

- **46(a):** We try solutions of the form

$$u(x,y) = f(x)g(y).$$

Inserting this $u(x,y)$ into the partial differential equation, we obtain

$$3f'(x)g(y) + 2f(x)g'(y) = 0,$$

or

$$\frac{3f'(x)}{f(x)} = \frac{-2g'(y)}{g(y)},$$

which can happen only if there is some constant λ such that

$$\frac{3f'(x)}{f(x)} = \lambda,$$

and

$$\frac{-2g'(y)}{g(y)} = \lambda.$$

It follows that

$$f(x) = Ae^{\lambda x/3},$$

and

$$g(y) = Be^{-\lambda y/2}.$$

So we have

$$u(x, y) = Ce^{\lambda((x/3)-(y/2))} = Ce^{k(2x-3y)},$$

so that

$$4e^{-x} = u(x, 0) = Ce^{2kx},$$

from which we conclude that $C = 4$ and $k = -\frac{1}{2}$. So the solution is

$$u(x, y) = 4e^{\frac{1}{2}(3y-2x)}.$$

- **53:** This problem is similar to Problem 12.19. The partial differential equation to be solved is

$$y_{tt} = a^2 y_{xx},$$

with

$$y(0, t) = y(2, t) = 0,$$

for all t,

$$y_t(x, 0) = 0,$$

and

$$y(x, 0) = f(x) = 0.03x(2 - x),$$

for all x in the interval $[0, 2]$.

We begin by seeking solutions of the form

$$y(x, t) = h(x)g(t).$$

Inserting this $y(x, t)$ into the partial differential equation, we get

$$h(x)g''(t) = a^2 h''(x)g(t),$$

from which it follows that

$$\frac{g''(t)}{a^2 g(t)} = -\lambda^2,$$

and

$$\frac{h''(x)}{h(x)} = -\lambda^2.$$

So we have

$$h(x) = A\cos(\lambda x) + B\sin(\lambda x),$$

and

$$g(t) = C\cos(a\lambda t) + D\sin(a\lambda t).$$

Now we use the constraints. Since we know that

$$y(0,t) = 0,$$

for all t, it follows that

$$h(0) = 0.$$

Then we must have

$$A\cos(\lambda 0) = 0,$$

or $A = 0$. Since

$$y(2,t) = 0,$$

for all t, we must also have

$$0 = h(2) = B\sin(2\lambda).$$

We don't want $B = 0$, since that would make $h(x) = 0$ for all x. So we select λ so that $\sin(2\lambda) = 0$; the possible choices are then

$$\lambda = \frac{m\pi}{2},$$

for any integer m. So far, we have found that the possible choices for $h(x)$ have the form

$$h(x) = B_m \sin(\frac{mx\pi}{2}),$$

for any integer m and constant B_m. From the constraint

$$y_t(x,0) = 0,$$

for all t, we have

$$0 = h(x)g'(0) = h(x)\Big(-a\lambda C\sin(a\lambda 0) + a\lambda D\cos(a\lambda 0)\Big).$$

Therefore, $D = 0$. So we have the choices

$$y(x,t) = K_m \sin(\frac{mx\pi}{2})\cos(\frac{amt\pi}{2}),$$

for arbitrary integer m and arbitrary constant K_m. Finally, we want to satisfy the constraint

$$y(x, 0) = f(x) = 0.03x(2 - x).$$

Therefore, we want to find the K_m so that

$$f(x) = 0.03x(2 - x) = \sum_{m=1}^{\infty} K_m \sin(\frac{mx\pi}{2}).$$

We must then find the Fourier sine coefficients for this function. We have done problems like this earlier, and I won't repeat the calculation here.