92.531 Applied Mathematics II: Solutions to Homework Problems in Chapter 13

- **13.55** (a) Since $|z+2-3i| = |z-(-2+3i)| = 5$, we want all the points whose distance from the point $(-2+3i)$ is 5. This is a circle with radius 5, centered at $(-2+3i)$.

  (b) Write $z = x+yi$, so that $|z+2|^2 = (x+2)^2+y^2$, and $|z-1|^2 = (x-1)^2+y^2$. Then we have
  $$(x+2)^2 + y^2 = 4\Big((x-1)^2 + y^2\Big),$$
  or
  $$x^2 + 4x + 4 + y^2 = 4\Big(x^2 - 2x + 1 + y^2\Big),$$
  so that
  $$x^2 + 4x + 4 + y^2 = 4x^2 - 8x + 4 + 4y^2.$$
  Then
  $$0 = 3x^2 - 12x + 3y^2 = 3\Big(x^2 - 4x + 4) + 3y^2 - 12,$$
  so that
  $$(x-2)^2 + y^2 = 4,$$
  which is a circle centered at $(2,0)$, with radius 2.

  (c) Write
  $$|z+5| = \Big(|z-5| + 6\Big)$$
  and square to get
  $$20x - 36 = 12\sqrt{(x-5)^2 + y^2}.$$
  Squaring both sides, we get
  $$256x^2 - 144y^2 - (144)(16) = 0,$$
  which reduces to
  $$\frac{x^2}{9} - \frac{y^2}{16} = 1,$$

which is a hyperbola. Notice that the original equation

$$|z+5|-|z-5|=|z-(-5)|-|z-(+5)|=6$$

tells us that the z we want are farther from -5 than they are from $+5$, so we want only the right half of the hyperbola, $x \geq 3$.

- **13.56** (a) Use the fact that

$$4 \leq |z-2+i|=|z-(2-i)|$$

to conclude that we want all z whose distance from the point $2-i$ is greater than or equal to 4, which is the boundary and exterior of a circle centered at $2-i$, with radius 4. Using $z=(x+iy)$, this is equivalent to the set of all points (x,y) with $(x-2)^2+(y+1)^2 \geq 16$.

(b) This is the segment of a circle of radius 3, centered at the origin, and bounded by a radius in the positive x direction and a radius at the 45 degree direction. Said another way, it is that part of the circular region $x^2+y^2 \leq 9$ bounded by the x-axis and the line $y=x$.

(c) An ellipse is the locus of points the sum of whose distances from two fixed points, the focal points, is a constant. Therefore,

$$|z+3|+|z-3|=|z-(-3)|+|z-(+3)|=10$$

is an ellipse with focal points $(3,0)$ and $(-3,0)$. The answer to the problem is then the interior of this ellipse. To solve the problem algebraically, we write

$$(x+3)^2+y^2=\left(10-\sqrt{(x-3)^2+y^2}\right)^2,$$

so that

$$x^2+6x+9+y^2=100-20\sqrt{(x-3)^2+y^2}+x^2-6x+9+y^2.$$

Then we have

$$20\sqrt{(x-3)^2+y^2}=100-12x.$$

Squaring both sides, we get

$$400\left(x^2-6x+9+y^2\right)=10,000-2400x+144x^2,$$

or

$$256x^2+400y^2=10,000-3600=6400$$

or

$$16x^2+25y^2=400.$$

- **13.57** (a) Use $z^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$.

  (b) Let $w = 3 + z$, so that $\frac{z}{3+z} = \frac{w-3}{w} = 1 - \frac{3}{w}$. Then let $w = a + ib$, so that $\frac{1}{w} = \frac{a-ib}{a^2+b^2}$.

  (c) With $z = x + iy$, we have $z^2 = (x^2 - y^2) + 2xyi$, so

  $$e^{z^2} = e^{(x^2-y^2)+2xyi} = e^{x^2-y^2}e^{2xyi} = e^{x^2-y^2}\Big(\cos(2xy) + i\sin(2xy)\Big).$$

  Therefore,

  $$e^{z^2} = e^{x^2-y^2}\cos(2xy) + ie^{x^2-y^2}\sin(2xy).$$

  (d) We write $z = x + iy$, so that

  $$1 + z = (x+1) + iy = \sqrt{(x+1)^2 + y^2}e^{i\theta},$$

  where $\tan\theta = \frac{y}{x+1}$. Then

  $$\ln(1+z) = \ln\left(\sqrt{(x+1)^2 + y^2}\right) + i\Big(\theta + 2k\pi\Big),$$

  for any integer k.

- **13.60** (a) and (b) The calculation is the same as it would be for the real function $x + \frac{1}{x}$, so the answer is $1 - z^{-2}$, which is not defined at $z = 0$.

- **13.62** The function $f(z) = x\sqrt{x^2+y^2} + iy\sqrt{x^2+y^2}$, and the Cauchy-Riemann equation $u_y = -v_x$ fails.

- **13.64** Let the analytic function be $f(z) = f(x+iy) = R(x,y) + iI(x,y)$, where $R(x,y)$ and $I(x,y)$ denote the real and imaginary parts of $f(z)$, respectively. We are given that $I(x,y) = 2x(1-y)$. Taking its partial derivatives, we find that $\frac{\partial I}{\partial x} = 2(1-y)$, and $\frac{\partial I}{\partial y} = -2x$. The Cauchy-Riemann equations tell us that

  $$\frac{\partial R}{\partial x} = \frac{\partial I}{\partial y} = -2x,$$

  and

  $$\frac{\partial R}{\partial y} = -\frac{\partial I}{\partial x} = 2(y-1).$$

  From

  $$\frac{\partial R}{\partial x} = -2x,$$

  it follows that $R(x,y) = -x^2 + g(y) + c$, for some real function $g(y)$ and real scalar c. From

  $$\frac{\partial R}{\partial y} = 2(y-1)$$

it follows that $g'(y) = 2y - 2$. Therefore, we know that $g(y) = y^2 - 2y$.

(a) Putting it all together, we get

$$R(x, y) = -x^2 + y^2 - 2y + c.$$

(b) Then we have

$$f(z) = -x^2 + y^2 - 2y + c + i(2x(1 - y)).$$

Substitute

$$x = \frac{z + \overline{z}}{2},$$

and

$$y = \frac{z - \overline{z}}{2i}$$

to get

$$f(z) = 2iz - z^2 + c.$$

- **13.65** The problem is incorrectly stated. The real part of $f(z)$ should read

$$R(x, y) = e^{-x}(x \cos y + y \sin y) + 1.$$

Then its first partial derivatives are

$$R_x(x, y) = -e^{-x}(x \cos y + y \sin y) + e^{-x}(\cos y) = e^{-x}((1 - x) \cos y - y \sin y),$$

and

$$R_y(x, y) = e^{-x}(-x \sin y + y \cos y + \sin y).$$

By the Cauchy-Riemann equations,

$$I_y(x, y) = R_x(x, y) = e^{-x}((1 - x) \cos y - y \sin y),$$

and

$$I_x(x, y) = -R_y(x, y) = e^{-x}(x \sin y - y \cos y - \sin y) = \frac{\partial}{\partial x}\Big(e^{-x}(y \cos y - x \sin y)\Big).$$

Integrating with respect to x, we get

$$I(x, y) = e^{-x}(y \cos y - x \sin y) + g(y) + c.$$

Taking the y partial, we get

$$I_y(x, y) = e^{-x}(\cos y - y \sin y - x \cos y) + g'(y) = e^{-x}((1 - x) \cos y - y \sin y) + g'(y).$$

But we also know that

$$I_y(x,y) = e^{-x}((1-x)\cos y - y\sin y),$$

from which we conclude that $g'(y) = 0$, so that $g(y) = d$, for some constant d. Therefore, we have

$$I(x,y) = e^{-x}(y\cos y - x\sin y) + k,$$

for some scalar k. So we have

$$f(z) = f(x+iy) = e^{-x}(x\cos y + y\sin y) + 1 + i[e^{-x}(y\cos y - x\sin y) + k].$$

From $f(0) = 1$, we find that $k = 0$. To put the function is terms of z, we note that $e^{-x}(\cos y - i\sin y) = e^{-z}$, so that

$$e^{-z}z = e^{-x}(\cos y - i\sin y)(x+iy) = e^{-x}(x\cos y + y\sin y) + i[e^{-x}(y\cos y - x\sin y)].$$

Therefore, $f(z) = ze^{-z} + 1$.

- **13.69** Since $2z+3$ is an analytic function, the integrals depend only on the end-points. An anti-derivative of $2z+3$ is $F(z) = z^2 + 3z$, so the integrals all have the value

$$F(3+i) - F(1-2i) = (3+i)^2 + 3(3+i) - (1-2i)^2 - 3(1-2i) = 17 + 19i.$$

In each of the integrals we have $z = x + iy$, so that

$$dz = \frac{dz}{dt}dt = (\frac{dx}{dt} + i\frac{dy}{dt})dt.$$

(a) Since $x(t) = 2t+1$, we have $x'(t) = 2$. Since $y(t) = 4t^2 - t - 2$, we have $y'(t) = 8t - 1$. Therefore,

$$dz = (2 + i(8t-1))dt.$$

Then we have

$$\int_{1-2i}^{3+i} 2z + 3dz = \int_0^1 (2x + 3 + 2iy)(2 + i(8t-1))dt$$

$$= \int_0^1 (2(2t+1) + 3 + 2i(4t^2 - t - 2))(2 + i(8t-1))dt$$

$$= \int_0^1 (-64t^3 + 24t^2 + 38t + 6) + i(48t^2 + 32t - 13)dt = 17 + 19i.$$

The other two parts of the problem are similar.

- **13.79** In (a) the denominator has a root at $z = -\frac{1}{2}$ that is repeated four times, and this is not a root of the numerator, so we have a pole of order four. In (b) the denominator has a single root at $z = 1$ and a double root at $z = -2$, and neither is a root of the numerator, so $z = 1$ is a simple pole, while $z = -2$ is a pole of order two. In (c), apply the quadratic formula to the denominator to get the roots $z = -1 + i$ and $z = -1 - i$. Neither is a root of the numerator, so they are simple poles. In (d), we write

$$\cos(\frac{1}{z}) = 1 - \frac{1}{2}z^{-2} + \frac{1}{24}z^{-4} - ...$$

and find that there are infinitely many negative powers of z involved. Therefore, $z = 0$ is an essential singularity. In (e), we expand the sine function in the numerator, to get

$$\sin(z - \frac{\pi}{3}) = (z - \frac{\pi}{3}) - \frac{1}{6}(z - \frac{\pi}{3})^3 + \frac{1}{120}(z - \frac{\pi}{3})^5 - ...,$$

so that, dividing by the denominator, we are left with

$$\frac{1}{3}\Big(1 - \frac{1}{6}(z - \frac{\pi}{3})^2 + \frac{1}{120}(z - \frac{\pi}{3})^4 - ...\Big).$$

This function is analytic everywhere, so the apparent singularity at $z = \frac{\pi}{3}$ is removable. Finally, in (f), the denominator has double roots at $z = 2i$ and $z = -2i$. These are not roots of the numerator, so both of them are double poles.

- **13.83** (a) The poles are $z = 2$ and $z = -2$. For the pole at $z = 2$ we let $t = z - 2$ and write

$$\frac{2z+3}{z+2} = \frac{2t+7}{t+4} = \frac{7}{4} + \frac{1}{16}t + ...,$$

so that, when we multiply by t^{-1}, the coefficient of the power t^{-1} is $\frac{7}{4}$, which is the residue. For the pole at $z = -2$, we let $t = z + 2$, so that

$$\frac{2z+3}{z-2} = \frac{2t-1}{t-4} = \frac{1}{4} + \frac{7}{4}t + ...$$

When we divide by t^{-1} the coefficient of t^{-1} becomes $\frac{1}{4}$, which is the residue.

(b) There is a double pole at $z = 0$ and a simple pole at $z = -5$. For the pole at $z = -5$, we set $t = z + 5$, so that

$$\frac{z-3}{z^2} = \frac{t-8}{(t-5)^2} = \frac{t-8}{t^2 - 10t + 25} = \frac{-8}{25} + \frac{-11}{125}t + ...$$

When we multiply by t^{-1} the coefficient of t^{-1} becomes $\frac{-8}{25}$, which is the residue. For the double pole at $z = 0$, we write

$$\frac{z-3}{z+5} = -\frac{3}{5} + \frac{8}{25}z - \frac{8}{125}z^2 + \ldots.$$

When we multiply by z^{-2}, the coefficient of the power z^{-1} becomes $\frac{8}{25}$, which is the residue.

For (c), we have a triple pole at $z = 2$. Since the letter t is already in use, we set $u = z - 2$, and write

$$e^{zt} = e^{(u+2)t} = e^{2t}e^{ut} = e^{2t}[1 + tu + \frac{1}{2}t^2u^2 + \ldots].$$

When we multiply by u^{-3} the coefficient of u^{-1} is $e^{2t}\frac{t^2}{2}$, which is the residue.

For (d), we have double poles at $z = i$ and $z = -i$. For the pole at $z = i$, we let $t = z - i$, so that

$$\frac{z}{(z+i)^2} = \frac{t+i}{(t+2i)^2} = \frac{t+i}{t^2+4it-4},$$

which we expand as

$$-\frac{i}{4} - \frac{i}{16}t^2 + \ldots.$$

When we multiply by t^{-2}, the coefficient of t^{-1} is zero, which is the residue. The calculations for the pole at $z = -i$ are similar.

- **13.86** To get the residue at $z = i$ we differentiate the function $ze^{tz}/(z+i)^2$ to get

$$\Big((z+i)^2(zte^{tz} + e^{tz}) - 2ze^{tz}(z+i)\Big)/(z+i)^4.$$

Taking the limit as $z \to i$, we get the residue at $z = i$, which is

$$\frac{-it}{4}e^{it}.$$

Similarly, the residue at $z = -i$ is $\frac{it}{4}e^{-it}$. The sum of these two residues is $\frac{t}{2}\sin t$. To get the integral, we must multiply this by $2\pi i$.

- **13.88** We consider the integral of the function

$$f(z) = \frac{z^2}{z^4+1}$$

around the contour consisting of the interval $[-R, R]$, followed by the upper semi-circle centered at the origin, with radius R. As $R \to \infty$, the integral over the semi-circle will go to zero, leaving us with the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx,$$

which is twice the integral in the problem.

Inside this contour, there are two poles,

$$z = u = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

and

$$z = v = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i.$$

The denominator factors as

$$z^4 + 1 = (z+u)(z-u)(z+v)(z-v).$$

We calculate the residue at $z = u$. We have

$$\frac{z^2}{(z+u)(z+v)(z-v)} = \frac{z^2}{(z+u)(z^2+i)} = \frac{z^2}{z^3 + uz^2 + iz + iu}.$$

We substitute $t = z - u$, or $z = t + u$, to get

$$\frac{(t+u)^2}{(t+u)^3 + u(t+u)^2 + i(t+u) + iu}.$$

This is an analytic function in a neighborhood of $z = u$, and can be expanded as a Taylor series with respect to the variable t; we want the constant term, since that term will be the coefficient of t^{-1} when we multiply by $(z-u)^{-1} = t^{-1}$. The constant term is easily seen to be $1/4u$, which is the residue at $z = u$. The residue at $z = v$, which is $1/4v$, is found in similar fashion. The sum of these residues is

$$\frac{1}{4u} + \frac{1}{4v} = -\frac{\sqrt{2}}{4} i.$$

The integral around the contour is then

$$(2\pi i)(-\frac{\sqrt{2}}{4} i) = \frac{\sqrt{2}\pi}{2}.$$

The desired answer is half of this, or

$$\frac{\sqrt{2}\pi}{4} = \frac{\pi}{2\sqrt{2}}.$$