92.530 Applied Mathematics I: Solutions to Homework Problems in Chapter 15

- **49.** We need to solve the system

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ -1 \end{bmatrix}.$$

The inverse of the 2 by 2 matrix

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

is

$$\frac{1}{5}\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix},$$

so

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -8 \\ -1 \end{bmatrix} = \begin{bmatrix} -33/5 \\ -26/5 \end{bmatrix}.$$

- **50:** Take

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

It is also possible for non-square matrices; take

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- **51:** Note that with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix},$$

so c and d can be anything.

- **55:** The point $(1,0)$ in the (x',y') system is the point $(\cos\theta, \sin\theta)$ in the original system, and the point $(0,1)$ in the (x',y') system is the point $(-\sin\theta, \cos\theta)$ in the original system. Therefore, the point

$$(x',y') = x'(1,0) + y'(0,1)$$

in the (x',y') system is the point

$$(x,y) = (x'\cos\theta - y'\sin\theta, x'\sin\theta + y'\cos\theta)$$

in the original system. Therefore, the point (x',y') in the second system is the point (x,y) in the original system, where

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Therefore, using the formula for the inverse of a 2 by 2 matrix, we get

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Note that the book is incorrect on this point. Also, the assertion in part (b) is wrong.

- **59:** (a) The matrix A is

$$A = \begin{bmatrix} 4 & -3 \\ -3 & 3 \end{bmatrix}.$$

(b) The matrix A is

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 2 \\ -3 & 2 & 4 \end{bmatrix}.$$

- **60:** Denote by $A^\dagger$ the conjugate transform of A, that is,

$$A^\dagger = \overline{A}^T.$$

(a) If $A^\dagger = A$, the complex conjugate of the number $X^\dagger AX$ is

$$(X^\dagger AX)^\dagger = X^\dagger A^\dagger (X^\dagger)^\dagger = X^\dagger AX,$$

so the number $X^\dagger AX$ must be a real number. (b) On the other hand, we see, by a similar argument, that if $A^\dagger = -A$ then the complex conjugate of $X^\dagger AX$ is its negative, so $X^\dagger AX$ must be purely imaginary or zero.

- **61:** Let $A = \frac{1}{2}(C + C^\dagger)$ and $B = \frac{1}{2}(C - C^\dagger$.

- **87:** First, find (x, y, z) so that

$$2x - 3y + z = 0,$$

and

$$-2x - y + z = 0.$$

The second equation tells us that

$$z = 2x + y.$$

Inserting this into the first equation, we find

$$0 = 2x - 3y + 2x + y = 4x - 2y,$$

or $y = 2x$. Therefore, $z = 4x$. The vectors $(x, 2x, 4x)$ are then orthogonal to both of the other two given vectors. To make this vector a unit vector, we want its length squared, which is $21x^2$, to be one, or $x = 1/\sqrt{21}$.

- **93:** For example, the second equation tells us that

$$x_3 = -1 - 2x_1 + 3x_2.$$

Inserting this into the first equation, we get

$$2 = 3x_1 + 2x_2 - 4(-1 - 2x_1 + 3x_2) = 11x_1 - 10x_2 + 4,$$

so that

$$x_2 = (11x_1 + 2)/10.$$

It follows that

$$x_3 = -1 - 2x_1 + 3((11x_1 + 2)/10).$$

So we have both x_2 and x_3 in terms of x_1. Selecting any value for x_1 gives us a solution.

- **96(a):** We begin by taking the determinant of the matrix $A - \lambda I$, where A is the matrix

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}.$$

Then

$$\det \begin{bmatrix} 2-\lambda & 2 \\ -1 & 5-\lambda \end{bmatrix} = \lambda^2 - 7\lambda + 12 = (\lambda - 4)(\lambda - 3).$$

Therefore, the eigenvalues are $\lambda = 3$ and $\lambda = 4$.

- **98:** If λ is an eigenvalue of the matrix A, then there is a non-zero vector u with $Au = \lambda u$. Then we also have

$$A^2u = A(Au) = A(\lambda u) = \lambda(Au) = \lambda(\lambda u) = \lambda^2 u.$$

More generally, $A^n u = \lambda^n u$, for any positive integer n. If A is invertible, then no eigenvalue of A can be zero, and we also have $A^{-1}u = \lambda^{-1}u$, as well as $A^n u = \lambda^n u$, for any integer n.

- **99** Suppose that the matrix A is skew-Hermitian, so that $A^\dagger = -A$. Then consider the equation $Au = \lambda u$, from which we can write

$$u^\dagger A u = \lambda u^\dagger u.$$

From Problem 15.58(c), we know that the quadratic form $u^\dagger A u$ is purely imaginary or zero. Since $u^\dagger u > 0$, it follows that λ is purely imaginary or zero.

- **107:** (a) We want to use Theorem 15-16. The quadratic form $x^2 + xy + y^2$ can be written as

$$x^2 + xy + y^2 = [\,x \quad y\,] \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = X^T A X,$$

for A the symmetric matrix

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

and

$$X = [x, y]^T.$$

The matrix A has eigenvalues $\lambda_1 = \frac{3}{2}$ and $\lambda_2 = \frac{1}{2}$, with associated normalized eigenvectors

$$u^1 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$$

and

$$u^2 = [\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T,$$

respectively. Let U be the matrix with u^1 as its first column and u^2 as its second column. Then $U^{-1} = U^T$ and Theorem 15-16 simplifies quite a bit. Let L be the diagonal matrix with the entries λ_1 and λ_2 on the main diagonal. Then $AU = UL$, and $A = ULU^T$. The quadratic form $X^T A X$ is then

$$X^T A X = X^T (ULU^T) X = (L^{1/2} U^T X)^T (L^{1/2} U^T X).$$

Therefore, we take

$$X' = [x', y']^T = L^{1/2}U^T X,$$

so that

$$x' = \frac{\sqrt{3}}{2}(x + y),$$

and

$$y' = \frac{1}{2}(-x + y).$$

(b) The matrix U is

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix},$$

so U has the form

$$U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

for $\theta = \frac{\pi}{4}$. This corresponds to a rotation of the axes through an angle of $\frac{-\pi}{4}$. Then the multiplication by

$$L^{1/2} = \begin{bmatrix} \sqrt{\frac{3}{2}} & 0 \\ 0 & \sqrt{\frac{1}{2}} \end{bmatrix}$$

means that the unit length along each of the new coordinate axes is rescaled. The overall effect is to turn a circle into an ellipse with its major axis along the line $y = x$.

- **108:** Suppose we want to maximize or minimize the function $f(x, y)$ given by

$$f(x, y) = x^2 + y^2,$$

subject to the condition

$$g(x, y) = x^2 + xy + y^2 = 16.$$

The method of using Lagrange multipliers involves setting to zero the first partial derivatives of the Lagrangian function

$$L(x, y; \alpha) = f(x, y) + \alpha g(x, y).$$

Then we have

$$0 = L_x(x, y; \alpha) = 2x + 2\alpha x + \alpha y,$$

and

$$0 = L_y(x, y; \alpha) = 2y + 2\alpha y + \alpha x.$$

Using $\lambda = -1/\alpha$, we can write these equations as

$$\lambda x = x + \frac{1}{2}y,$$

and

$$\lambda y = \frac{1}{2}x + y,$$

so that

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, the solution vector $X = [x, y]^T$ is an eigenvector of the matrix

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}.$$

From

$$16 = x^2 + xy + y^2 = X^T AX = \lambda X^T X = \lambda(x^2 + y^2),$$

we find that the corresponding values of $f(x, y)$ are $16/\lambda$, for the two eigenvalues of A, $\lambda_1 = \frac{3}{2}$ and $\lambda_2 = \frac{1}{2}$. Choosing λ_1 gives the minimum value of $f(x, y)$, and choosing λ_2 gives the maximum value.