92.531 Applied Mathematics II: Solutions to Homework Problems in Chapter 16

- **16.26** From the Euler-Lagrange equation we know that

$$\frac{d}{dx}(\frac{\partial f}{\partial y'}) = 0,$$

which tells us that

$$\frac{\partial f}{\partial y'} = c,$$

for some constant c.

- **16.27** In this case we have

$$f(x, y, y') = \sqrt{1 + (y')^2},$$

so that

$$\frac{\partial f}{\partial y'} = y'/\sqrt{1 + (y')^2}.$$

From the previous problem, we have

$$\frac{\partial f}{\partial y'} = y'/\sqrt{1 + (y')^2} = c,$$

or

$$y' = c\sqrt{1 + (y')^2},$$

which tells us that y' has constant sign. Also, c cannot be 1. Squaring, and solving, we get

$$(y')^2 = c/(1 - c).$$

Since y' has constant sign, it follows that $y'(x) = \sqrt{c/(1 - c)}$, so that $y(x)$ is linear.

- **16.29** With $f(u, v, w) = (1 + w^2)^{1/2} v^{-1/2}$, we have

$$f_v = (1 + w^2)^{1/2} - \frac{1}{2} v^{-3/2},$$

and

$$f_w = \frac{w}{\sqrt{v}\sqrt{1+w^2}}.$$

Then

$$\frac{d}{dx} f_{y'} = y^{-1/2}(1+y'^2)^{-1/2}y'' - y^{-1/2}y'^2(1+y'^2)^{-3/2} - \frac{1}{2}y^{-3/2}y'^2(1+y'^2)^{-1/2}.$$

From Euler's Equation and a bit of algebra we get

$$0 = (1+y'^2)^2 + 2yy'' - y'^2(1+y'^2) = 1 + y'^2 + 2yy''.$$

The answer in the book has a minus sign in front of the $2yy''$, but I think it is wrong.

- **16.33** We have $f(u,v,w) = w^2 - v^2$, so that $f_w = 2w$. Since $f_u = 0$, we have

$$c = f - y' f_{y'} = y'^2 - y^2 - y'(2y') = -y'^2 - y^2,$$

so that

$$y^2 + y'^2 = -c.$$

Differentiating, we get

$$2yy' + 2y'y'' = 0,$$

so that

$$y'' + y = 0,$$

or

$$y' = 0.$$

Therefore, $y(x) = B \sin x$, or $y(x) = 0$. The book's answer is not correct.

- **16.40** At the end of the solution to Problem 16.5 the author notes that he has not proved that the surface is actually a minimum. The curves of the form

$$y = c \cosh\Big(\frac{x+k}{c}\Big)$$

must go through the point (x_1, y_1), which will reduce the family of curves to a one-parameter family. The interested student may try to plot members of the family for fixed values of one of the parameters and the other varying. For an arbitrary second point (x_2, y_2), there may be one, two or no member of the family passing through this point.

Problem 16.40 asks for a curve of fixed length that minimizes the surface area. Using Lagrange multiplier λ, we find that the relevant integrand is

$$f = y\sqrt{1+(y')^2} + \lambda\sqrt{1+(y')^2}.$$

This f can be rewritten as

$$f = (y+\lambda)\sqrt{1+((y+\lambda)')^2},$$

which means that the function $y(x)+\lambda$ must have the form of the solution to Problem 16.5. Now we have two points that must lie on the curve, as well as a specified length for the curve. Again, it may not be possible to find a solution of the desired form. It could be the case, if the length of the curve is given to be $y_1 + (x_2 - x_1) + y_2$, that the minimizing curve consists of three lines: the vertical line from (x_1, y_1) to $(x_1, 0)$, the horizontal line from $(x_1, 0)$ to $(x_2, 0)$, and the vertical line from $(x_2, 0)$ to (x_2, y_2).