92.530 Applied Mathematics I: Solutions to Homework Problems in Chapter 1

- **1.70** The first two parts follow immediately from the definitions. (c): We have

$$\sinh(\log(x+\sqrt{x^2+1})) = \frac{1}{2}\Big(\exp(\log(x+\sqrt{x^2+1})) - \exp(-\log(x+\sqrt{x^2+1}))\Big)$$

$$= \frac{1}{2}\Big(x+\sqrt{x^2+1} - \frac{1}{x+\sqrt{x^2+1}}\Big) = \frac{1}{2}\frac{2x^2+2x\sqrt{x^2+1}+1-1}{x+\sqrt{x^2+1}}$$

$$= \frac{1}{2}\frac{2x(x+\sqrt{x^2+1})}{x+\sqrt{x^2+1}} = \frac{2x}{x} = x.$$

- **1.130** We have

$$\frac{\partial f}{\partial x} = 2x\tan^{-1}\frac{y}{x} + x^2\frac{-yx^{-2}}{1+(\frac{y}{x})^2} = 2x\tan^{-1}\frac{y}{x} - \frac{x^2 y}{x^2+y^2},$$

and

$$\frac{\partial f}{\partial y} = x^2\frac{x^{-1}}{1+(\frac{y}{x})^2} = \frac{x^3}{x^2+y^2}.$$

Therefore,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2x^2\tan^{-1}\frac{y}{x} - \frac{x^3 y}{x^2+y^2} + y\frac{x^3}{x^2+y^2} = 2f.$$

- **1.131** We calculate $\frac{\partial V}{\partial x}$ and $\frac{\partial^2 V}{\partial x^2}$; the others are similar. We have

$$\frac{\partial V}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2},$$

and

$$\frac{\partial^2 V}{\partial x^2} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}.$$

The other two second partial derivatives are similar. Their sum is zero.

- **1.134** We have

$$\frac{\partial z}{\partial x} = 2xf(\frac{y}{x}) + x^2 f'(\frac{y}{x})(\frac{-y}{x^2}),$$

and

$$\frac{\partial z}{\partial y} = x^2 f'(\frac{y}{x})(\frac{1}{x}).$$

Therefore,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} =$$

$$x\Big(2xf(\frac{y}{x}) + x^2 f'(\frac{y}{x})(\frac{-y}{x^2})\Big) + y\Big(x^2 f'(\frac{y}{x})(\frac{1}{x})\Big)$$

$$= 2x^2 f(\frac{y}{x}) = 2z.$$

- **1.135** We calculate the right side of the equation and show that it equals the left side. For simplicity denote $\cos\phi$ by c, and $\sin\phi$ by s.

$$u_\rho = u_x x_\rho + u_y y_\rho = u_x c + u_y s.$$

Then, because

$$c_\rho = s_\rho = 0,$$

we have

$$u_{\rho\rho} = u_{x\rho}c + u_{y\rho}s = (u_{xx}c + u_{xy}s)c + (u_{yx}c + u_{yy}s)s.$$

Also,

$$u_\phi = u_x x_\phi + u_y y_\phi = u_x(-\rho s) + u_y(\rho c).$$

So,

$$u_{\phi\phi} = (-\rho u_x)c + (-\rho u_x)_\phi s - (\rho u_y)s + (\rho u_y)_\phi c$$

$$= -\rho u_x c - \rho(u_{xx}x_\phi + u_{xy}y_\phi)s$$

$$-\rho u_y s + \rho(u_{yx}x_\phi + u_{yy}y_\phi)c.$$

Therefore, we have

$$u_{\phi\phi} = -\rho(u_x c + u_y s) + \rho^2 s^2 u_{xx} - 2\rho^2 cs u_{xy} + \rho^2 c^2 u_{yy}.$$

Therefore, we have

$$u_{\rho\rho} = c^2 u_{xx} + 2cs u_{xy} + s^2 u_{yy},$$

$$\frac{1}{\rho}u_\rho = \frac{1}{\rho}(cu_x + su_y),$$

and

$$\frac{1}{\rho^2}u_{\phi\phi} = -\frac{1}{\rho}(cu_x + su_y) + s^2 u_{xx} - 2cs u_{xy} + c^2 u_{yy}.$$

Using $c^2 + s^2 = 1$, it follows easily that

$$u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi} = u_{xx} + u_{yy}.$$

- **1.137** For $f(x, y, z) = xz + y^2$, we have

$$\frac{\partial f}{\partial x} = z,$$

$$\frac{\partial f}{\partial y} = 2y,$$

$$\frac{\partial f}{\partial z} = x,$$

$$\frac{\partial^2 f}{\partial z \partial x} = 1,$$

$$\frac{\partial^2 f}{\partial y^2} = 2,$$

$$\frac{\partial^2 f}{\partial x \partial z} = 1,$$

and all others are zero. At the point $(1, -1, 2)$ we have

$$f(1, -1, 2) = 3,$$

$$\frac{\partial f}{\partial x}(1, -1, 2) = 2,$$

$$\frac{\partial f}{\partial y}(1, -1, 2) = -2,$$

$$\frac{\partial f}{\partial z}(1, -1, 2) = 1,$$

$$\frac{\partial^2 f}{\partial z \partial x}(1, -1, 2) = 1,$$

$$\frac{\partial^2 f}{\partial y^2}(1, -1, 2) = 2,$$

and

$$\frac{\partial^2 f}{\partial x \partial z}(1, -1, 2) = 1.$$

The Taylor series is then

$$3+2(x-1)-2(y+1)+1(z-2)+\frac{1}{2}((1)(x-1)(z-2)+(2)(y+1)^2+(1)(z-2)(x-1)).$$

Expanding this, we get

$$3+2x-2-2y-2+z-2+(x-1)(z-2)+(y+1)^2 = 3+2x-2-2y-2+z-2+xz-2x-z+2+y^2+2y+1$$

$$= xz + y^2.$$

- **1.138** We begin with the array

$$\begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & -2 & -2 & 4 \\ 3 & -1 & 4 & 2 \end{bmatrix}.$$

For the first step, we switch rows one and two, obtaining

$$\begin{bmatrix} 1 & -2 & -2 & 4 \\ 2 & 1 & 1 & 3 \\ 3 & -1 & 4 & 2 \end{bmatrix}.$$

Adding -2 times the first row to the second row, adding -3 times the first row to the third row, and then dividing both the second and third rows by 5, the array becomes

$$\begin{bmatrix} 1 & -2 & -2 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & -2 \end{bmatrix}.$$

Adding -1 times the second row to the third row, we get

$$\begin{bmatrix} 1 & -2 & -2 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

This array corresponds to the system of equations

$$x - 2y - 2z = 4,$$

$$y + z = -1,$$

$$z = -1,$$

which has the solution $z = -1$, $y = 0$, and $x = 2$.

- **1.148** We want to optimize the function $f(x, y) = x^2 + y^2$, subject to the constraint $x^2 + xy + y^2 - 16 = 0$. The Lagrangian is

$$h(x, y) = x^2 + y^2 + \lambda(x^2 + xy + y^2 - 16).$$

Then,

$$\frac{\partial h}{\partial x} = 2x + \lambda(2x + y) = 0,$$

and

$$\frac{\partial h}{\partial y} = 2y + \lambda(x + 2y) = 0.$$

Solving for λ in both equations, we get

$$\lambda = \frac{-2x}{2x+y} = \frac{-2y}{x+2y}.$$

From the last equation we have

$$\frac{x}{2x+y} = \frac{y}{x+2y},$$

or

$$x(x+2y) = y(2x+y),$$

so that

$$x^2 + 2xy = 2xy + y^2,$$

or

$$x^2 = y^2.$$

Using $y = x$, the equation

$$x^2 + xy + y^2 - 16 = 0$$

becomes

$$3x^2 = 16.$$

The solutions then are the points $(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}})$ and $(\frac{-4}{\sqrt{3}}, \frac{-4}{\sqrt{3}})$; the distance to the origin is then $\frac{4\sqrt{6}}{3}$. Using

$$y = -x,$$

the equation

$$x^2 + xy + y^2 - 16 = 0$$

becomes

$$x^2 = 16,$$

so the solutions are the points $(4, -4)$ and $(-4, 4)$; the distance to the origin is then $4\sqrt{2}$. The curve described by the constraint function is an ellipse. The points closest to the origin are at the two ends of its minor axis, while the points farthest away are at the two ends of its major axis.

- **1.151** (a) The integral is

$$\int_{\sin\alpha}^{\cos\alpha} (x^2 \sin\alpha - x^3)dx = \frac{1}{3}[\cos^3\alpha \sin\alpha - \sin^4\alpha] - \frac{1}{4}[\cos^4\alpha - \sin^4\alpha].$$

Differentiating, we get

$$\frac{1}{3}[-3\cos^2\alpha\sin^2\alpha + \cos^4\alpha - 4\sin^3\alpha\cos\alpha] - \frac{1}{4}[-4\cos^3\alpha\sin\alpha - 4\sin^3\alpha\cos\alpha].$$

(b): According to Leibniz's Rule, the derivative is

$$\int_{\sin\alpha}^{\cos\alpha} x^2\cos\alpha\,dx + (\cos^2\alpha\sin\alpha - \cos^3\alpha)(-\sin\alpha) - (\sin^3\alpha - \sin^3\alpha)(\cos\alpha).$$

The integral is

$$\int_{\sin\alpha}^{\cos\alpha} x^2\cos\alpha\,dx = \frac{1}{3}[\cos^4\alpha - \sin^3\alpha\cos\alpha].$$

The rest is algebra.

- **1.160** (a): $\tan\theta = \frac{3}{3\sqrt{3}}$ so that $\tan\theta = \frac{1}{\sqrt{3}}$ and $\theta = \frac{\pi}{6}$. The magnitude is $\sqrt{36} = 6$, so

  $$3\sqrt{3} + 3i = 6\exp(\frac{i\pi}{6}) = 6\,\mathrm{cis}\,(\frac{\pi}{6}).$$

  The others are done similarly.

- **1.162** (a): The angle is $\theta = \frac{\pi}{4}$ and the magnitude is 8, so the number can be written as $z = 8\exp(\frac{\pi}{4})$. The third roots of z are then $2\exp(\frac{\pi}{12})$, $2\exp(\frac{\pi}{12} + \frac{2\pi}{3})$, and $2\exp(\frac{\pi}{12} + \frac{4\pi}{3})$. The others are done similarly.