• **2.54** (a): Differentiating $y$, we have

$$y' = e^{-x}(-c_1 \sin x + c_2 \cos x) - e^{-x}(c_1 \cos x + c_2 \sin x),$$

so that

$$y' = e^{-x}(-c_1(\sin x + \cos x) + c_2(\cos x - \sin x)).$$

Then,

$$y'' = e^{-x}(-c_1(\cos x - \sin x) + c_2(-\sin x - \cos x)) - e^{-x}(-c_1(\sin x + \cos x) + c_2(\cos x - \sin x),$$

or

$$y'' = e^{-x}(c_1(-\cos x + \sin x + \sin x + \cos x) + c_2(-\sin x - \cos x - \cos x + \sin x))$$

so

$$y'' = e^{-x}(2c_1 \sin x - 2c_2 \cos x).$$

Clearly, $y$ solves the differential equation. (b): From $y(0) = -2$, we have $c_1 = -2$. From $y'(0) = 5$, we have $-c_1 + c_2 = 2 + c_2 = 5$, or $c_2 = 3$. So the particular solution is $y = e^{-x}(3 \sin x - 2 \cos x)$.

• **2.55** For part (a), just differentiate. (b): The first family consists of the general solutions, consisting of straight lines; the second is a singular solution that is a quadratic. (c) The members of the family of general solutions are lines tangent to the quadratic solution, and the latter provides an *envelope* for the family.

• **2.83** Let $(a, f(a))$ be a fixed point on the unknown curve. The line normal to this curve at this point is

$$y = -\frac{1}{f'(a)}(x - a) + f(a),$$
while the line through the origin that is perpendicular to this normal line is

\[ y = f'(a)x. \]

These two lines intersect at the point

\[(x, y) = \left( \frac{a + f'(a)f(a)}{1 + f'(a)^2}, \frac{af'(a) + f(a)f'(a)^2}{1 + f'(a)^2} \right).\]

The square of the distance from this point to the origin is

\[ \frac{(a + f'(a)f(a))^2}{1 + f'(a)^2}, \]

which, we are told, must equal \( f(a)^2 \). Therefore, we have

\[ a^2 + 2af'(a)f(a) + f'(a)^2f(a)^2 = f'(a)^2 + f'(a)^2f(a)^2, \]

or

\[ a^2 + 2af'(a)f(a) = f(a)^2. \]

This holds for all points \((a, f(a))\) on the curve. Rewriting this as

\[ \frac{2af'(a)f(a) - f(a)^2}{a^2} = -1, \]

we have

\[ \frac{d}{da} \frac{f(a)^2}{a} = -1, \]

so that

\[ \frac{f(a)^2}{a} = -a + c, \]

for some \( c \), which we can easily show must be 5. Finally, we have

\[ f(a)^2 + a^2 = 5a, \]

or, in the more familiar notation, \( x^2 + y^2 = 5x \).

\[ \textbf{3.54 (a):} \] We have

\[ (0)x^2 + \frac{7}{5}(3x + 2) - \frac{11}{5}(x - 1) - (1)(2x + 5) = 0, \]

for all \( x \), so the four functions are linearly dependent. (b): If there are constants \( a, b, c \), such that

\[ ax^2 + b(3x + 2) + c(x - 1) = 0, \]

for all \( x \), then, differentiating twice, we get \( 2a = 0 \), so \( a = 0 \). Then

\[ b(3x + 2) + c(x - 1) = 0, \]

for all \( x \). Taking \( x = 1 \), we get \( b = 0 \), and taking \( x = \frac{2}{3} \), we get \( c = 0 \). So the three functions are linearly independent.
• 3.55 Suppose that

\[ ae^x + bxe^x + cx^2e^x = 0, \]

for all \( x \). Taking \( x = 0 \), we get that \( a = 0 \). Therefore,

\[ bxe^x + cx^2e^x = 0, \]

for all \( x \). Taking \( x = 1 \), we get \( be + ce = 0 \), while taking \( x = -1 \), we get \( -be + ce = 0 \). It follows that both \( b \) and \( c \) equal zero, and the three functions are linearly independent.

• 3.56 We use the idea described in I.2. on page 76. Since \( y = x \) is a known solution, we look for solutions of the form \( y = vx \), for some \( v \) to be found. Substituting \( y = vx \) into the differential equation, we get

\[ x^2v'' + (2x + x^2)v = 0. \]

Setting \( u = v' \), and restricting \( x \) to positive values, we have

\[ \frac{du}{dx} = -\frac{2x + x^2}{x^2}u. \]

Separating variables, we find that

\[ \frac{du}{u} = (-1 - \frac{2}{x})dx. \]

Integrating, we obtain

\[ \log |u| = -x - 2(\log x) + C, \]

or

\[ u = Ke^{-x}(\frac{1}{x^2}), \]

for some constant \( K \). Since \( u = v' \), it follows that

\[ v = K \int \frac{e^{-x}}{x^2} dx + A, \]

for some constant \( A \), and finally

\[ y = vx = Ax + Kx \int \frac{e^{-x}}{x^2} dx. \]

We leave the indefinite integral in the answer, since we cannot find the anti-derivative in closed form.
3.76 Differentiating both sides of the first equation, we get

\[ x'' + y' = e^t. \]

Solving for \( y' \) and substituting into the second equation, we get

\[ t = x - y' = x + x'' - e^t, \]

or

\[ x'' + x = t + e^t, \]

which is a second-order linear non-homogeneous ODE, with constant coefficients. Using operator notation, we have

\[(D^2 + I)x = t + e^t.\]

The auxiliary equation for the homogeneous problem is

\[ m^2 + 1 = 0, \]

with solutions \( m = i \) and \( m = -i \). Therefore, the general solution of the homogeneous problem

\[ x'' + x = 0 \]

is

\[ x(t) = c_1 \cos t + c_2 \sin t. \]

To solve the nonhomogeneous problem, we proceed as in problem 3.18, using the method of undetermined coefficients. We seek a solution of the form

\[ x(t) = at + b + ce^t. \]

Substituting this in the original problem,

\[ x'' + x = t + e^t, \]

we get

\[ ce^t + at + b + ce^t = t + e^t, \]

so that \( a = 1, b = 0 \) and \( c = \frac{1}{2} \). The general solution for \( x(t) \) is

\[ x(t) = c_1 \cos t + c_2 \sin t + t + c \frac{1}{2} e^t. \]
To find $y(t)$, we use the first of the two original differential equations,

$$\frac{dx}{dt} + y = e^t,$$

which tells us that

$$y(t) = e^t - x'(t) = e^t - (c_2 \cos t - c_1 \sin t) - 1 - \frac{1}{2}e^t$$

$$= (c_1 \sin t - c_2 \cos t) + \frac{1}{2}e^t - 1.$$

Note that in the book's answer, $c_2$ is called $-c_2$.

**3.83** The differential equation is

$$x'' + 4x = 8 \sin \omega t.$$

The auxiliary equation for the homogeneous problem is

$$m^2 + 4 = 0,$$

with solutions $m = 2i$ and $m = -2i$. Therefore, the general solution of the homogeneous problem is

$$x(t) = c_1 \cos 2t + c_2 \sin 2t.$$

If $\omega$ is not equal to 2 or $-2$, then we can find the general solution to the nonhomogeneous problem by seeking a solution of the form

$$x(t) = a \cos \omega t + b \sin \omega t.$$

Substituting, we get

$$-\omega^2a \cos \omega t - \omega^2b \sin \omega t + 4(a \cos \omega t + b \sin \omega t) = 8 \sin \omega t.$$

It follows that

$$4a - \omega^2 a = 0$$

and

$$4b - \omega^2 b = 8.$$

Therefore, $a = 0$ and $b = \frac{8}{4 - \omega^2}$. The general solution is then

$$x(t) = c_1 \cos 2t + c_2 \sin 2t + 8 \frac{8}{4 - \omega^2} \sin \omega t.$$
Since \( x(0) = 0 \), we know that \( c_1 = 0 \). From
\[
x'(t) = 2c_2 \cos 2t + \frac{8}{4 - \omega^2} \omega \cos \omega t,
\]
and \( x'(0) = 0 \), we get
\[
2c_2 + \frac{8}{4 - \omega^2} \omega = 0,
\]
so that
\[
c_2 = \frac{4\omega}{\omega^2 - 4}.
\]
The answer, therefore, is
\[
x(t) = \frac{4\omega}{\omega^2 - 4} \sin 2t + \frac{8}{4 - \omega^2} \sin \omega t,
\]
or
\[
x(t) = \frac{8 \sin \omega t - 4\omega \sin 2t}{4 - \omega^2}.
\]
Now we consider the case of \( \omega^2 = 4 \).
If \( \omega = 2 \), then we seek a solution of the nonhomogeneous problem having the form
\[
x(t) = at \cos 2t + bt \sin 2t.
\]
Note that if \( \omega = -2 \), our trial solution would have the same form. Differentiating, we get
\[
x'(t) = -2at \sin 2t + a \cos 2t + 2bt \cos 2t + b \sin 2t = (a + 2bt) \cos 2t + (b - 2at) \sin 2t.
\]
Differentiating again, we get
\[
x''(t) = -2(a + 2bt) \sin 2t + 2b \cos 2t + 2(b - 2at) \cos 2t - 2a \sin 2t = (4b - 4at) \cos 2t - (4a + 4bt) \sin 2t.
\]
Then
\[
x''(t) + 4x(t) = (4b - 4at) \cos 2t - (4a + 4bt) \sin 2t + 4(at \cos 2t + bt \sin 2t).
\]
It follows that
\[
4b \cos 2t - 4a \sin 2t = 8 \sin 2t,
\]
so that \( b = 0 \) and \( a = -2 \). The solution to the nonhomogeneous problem is then
\[
x(t) = -2t \cos 2t + c_1 \cos 2t + c_2 \sin 2t.
\]
from \( x(0) = 0 \), we get \( c_1 = 0 \). Since
\[
x'(t) = -4t \sin 2t - 2 \cos 2t + 2c_2 \cos 2t,
\]
and $x'(0) = 0$, we get $c_2 = 1$. Therefore, the particular solution is

$$x(t) = -2t \cos 2t + \sin 2t.$$ 

Resonance occurs when $\omega = 2$ or $\omega = -2$. 