92.530 Applied Mathematics I: Solutions to Homework Problems in Chapters 2 and 3

- **2.54** (a): Differentiating y, we have

$$y' = e^{-x}(-c_1 \sin x + c_2 \cos x) - e^{-x}(c_1 \cos x + c_2 \sin x),$$

so that

$$y' = e^{-x}(-c_1(\sin x + \cos x) + c_2(\cos x - \sin x)).$$

Then,

$$y'' = e^{-x}(-c_1(\cos x - \sin x) + c_2(-\sin x - \cos x)) - e^{-x}(-c_1(\sin x + \cos x) + c_2(\cos x - \sin x),$$

or

$$y'' = e^{-x}(c_1(-\cos x + \sin x + \sin x + \cos x) + c_2(-\sin x - \cos x - \cos x + \sin x))$$

so

$$y'' = e^{-x}(2c_1 \sin x - 2c_2 \cos x).$$

Clearly, y solves the differential equation. (b): From $y(0) = -2$, we have $c_1 = -2$. From $y'(0) = 5$, we have $-c_1 + c_2 = 2 + c_2 = 5$, or $c_2 = 3$. So the particular solution is $y = e^{-x}(3 \sin x - 2 \cos x)$.

- **2.55** For part (a), just differentiate. (b): The first family consists of the general solutions, consisting of straight lines; the second is a singular solution that is a quadratic. (c) The members of the family of general solutions are lines tangent to the quadratic solution, and the latter provides an *envelope* for the family.

- **2.83** Let $(a, f(a))$ be a fixed point on the unknown curve. The line normal to this curve at this point is

$$y = -\frac{1}{f'(a)}(x - a) + f(a),$$

while the line through the origin that is perpendicular to this normal line is

$$y = f'(a)x.$$

These two lines intersect at the point

$$(x, y) = (\frac{a + f'(a)f(a)}{1 + f'(a)^2}, \frac{af'(a) + f(a)f'(a)^2}{1 + f'(a)^2}).$$

The square of the distance from this point to the origin is

$$\frac{(a + f'(a)f(a))^2}{1 + f'(a)^2},$$

which, we are told, must equal $f(a)^2$. Therefore, we have

$$a^2 + 2af'(a)f(a) + f'(a)^2 f(a)^2 = f'(a)^2 + f'(a)^2 f(a)^2,$$

or

$$a^2 + 2af'(a)f(a) = f(a)^2.$$

This holds for all points $(a, f(a))$ on the curve. Rewriting this as

$$\frac{2af'(a)f(a) - f(a)^2}{a^2} = -1,$$

we have

$$\frac{d}{da}\frac{f(a)^2}{a} = -1,$$

so that

$$\frac{f(a)^2}{a} = -a + c,$$

for some c, which we can easily show must be 5. Finally, we have

$$f(a)^2 + a^2 = 5a,$$

or, in the more familiar notation, $x^2 + y^2 = 5x$.

- **3.54** (a): We have

$$(0)x^2 + \frac{7}{5}(3x + 2) - \frac{11}{5}(x - 1) - (1)(2x + 5) = 0,$$

for all x, so the four functions are linearly dependent. (b): If there are constants a, b, c, such that

$$ax^2 + b(3x + 2) + c(x - 1) = 0,$$

for all x, then, differentiating twice, we get $2a = 0$, so $a = 0$. Then

$$b(3x + 2) + c(x - 1) = 0,$$

for all x. Taking $x = 1$, we get $b = 0$, and taking $x = \frac{-2}{3}$, we get $c = 0$. So the three functions are linearly independent.

- **3.55** Suppose that

$$ae^x + bxe^x + cx^2e^x = 0,$$

for all x. Taking $x = 0$, we get that $a = 0$. Therefore,

$$bxe^x + cx^2e^x = 0,$$

for all x. Taking $x = 1$, we get $be + ce = 0$, while taking $x = -1$, we get $-be + ce = 0$. It follows that both b and c equal zero, and the three functions are linearly independent.

- **3.56** We use the idea described in I.2. on page 76. Since $y = x$ is a known solution, we look for solutions of the form $y = vx$, for some v to be found. Substituting $y = vx$ into the differential equation, we get

$$x^2v'' + (2x + x^2)v = 0.$$

Setting $u = v'$, and restricting x to positive values, we have

$$\frac{du}{dx} = -\frac{2x + x^2}{x^2}u.$$

Separating variables, we find that

$$\frac{du}{u} = (-1 - \frac{2}{x})dx.$$

Integrating, we obtain

$$\log|u| = -x - 2(\log x) + C,$$

or

$$u = Ke^{-x}(\frac{1}{x^2}),$$

for some constant K. Since $u = v'$, it follows that

$$v = K\int \frac{e^{-x}}{x^2}dx + A,$$

for some constant A, and finally

$$y = vx = Ax + Kx\int \frac{e^{-x}}{x^2}dx.$$

We leave the indefinite integral in the answer, since we cannot find the anti-derivative in closed form.

- **3.76** Differentiating both sides of the first equation, we get

$$x'' + y' = e^t.$$

Solving for y' and substituting into the second equation, we get

$$t = x - y' = x + x'' - e^t,$$

or

$$x'' + x = t + e^t,$$

which is a second-order linear non-homogeneous ODE, with constant coefficients. Using operator notation, we have

$$(D^2 + I)x = t + e^t.$$

The auxiliary equation for the homogeneous problem is

$$m^2 + 1 = 0,$$

with solutions $m = i$ and $m = -i$. Therefore, the general solution of the homogeneous problem

$$x'' + x = 0$$

is

$$x(t) = c_1 \cos t + c_2 \sin t.$$

To solve the nonhomogeneous problem, we proceed as in problem 3.18, using the method of undetermined coefficients. We seek a solution of the form

$$x(t) = at + b + ce^t.$$

Substituting this in the original problem,

$$x'' + x = t + e^t,$$

we get

$$ce^t + at + b + ce^t = t + e^t,$$

so that $a = 1$, $b = 0$ and $c = \frac{1}{2}$. The general solution for $x(t)$ is

$$x(t) = c_1 \cos t + c_2 \sin t + t + \frac{1}{2}e^t.$$

To find $y(t)$, we use the first of the two original differential equations,

$$\frac{dx}{dt} + y = e^t,$$

which tells us that

$$y(t) = e^t - x'(t) = e^t - (c_2 \cos t - c_1 \sin t) - 1 - \frac{1}{2}e^t$$

$$= (c_1 \sin t - c_2 \cos t) + \frac{1}{2}e^t - 1.$$

Note that in the book's answer, c_2 is called $-c_2$.

- **3.83** The differential equation is

$$x'' + 4x = 8 \sin \omega t.$$

The auxiliary equation for the homogeneous problem is

$$m^2 + 4 = 0,$$

with solutions $m = 2i$ and $m = -2i$. Therefore, the general solution of the homogeneous problem is

$$x(t) = c_1 \cos 2t + c_2 \sin 2t.$$

If ω is not equal to 2 or -2, then we can find the general solution to the nonhomogeneous problem by seeking a solution of the form

$$x(t) = a \cos \omega t + b \sin \omega t.$$

Substituting, we get

$$-\omega^2 a \cos \omega t - \omega^2 b \sin \omega t + 4(a \cos \omega t + b \sin \omega t) = 8 \sin \omega t.$$

It follows that

$$4a - \omega^2 a = 0$$

and

$$4b - \omega^2 b = 8.$$

Therefore, $a = 0$ and $b = \frac{8}{4-\omega^2}$. The general solution is then

$$x(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{8}{4 - \omega^2} \sin \omega t.$$

Since $x(0) = 0$, we know that $c_1 = 0$. From

$$x'(t) = 2c_2 \cos 2t + \frac{8}{4 - \omega^2} \omega \cos \omega t,$$

and $x'(0) = 0$, we get

$$2c_2 + \frac{8}{4 - \omega^2} \omega = 0,$$

so that

$$c_2 = \frac{4\omega}{\omega^2 - 4}.$$

The answer, therefore, is

$$x(t) = \frac{4\omega}{\omega^2 - 4} \sin 2t + \frac{8}{4 - \omega^2} \sin \omega t,$$

or

$$x(t) = \frac{8 \sin \omega t - 4\omega \sin 2t}{4 - \omega^2}.$$

Now we consider the case of $\omega^2 = 4$.

If $\omega = 2$, then we seek a solution of the nonhomogeneous problem having the form

$$x(t) = at \cos 2t + bt \sin 2t.$$

Note that if $\omega = -2$, our trial solution would have the same form. Differentiating, we get

$$x'(t) = -2at \sin 2t + a \cos 2t + 2bt \cos 2t + b \sin 2t = (a + 2bt) \cos 2t + (b - 2at) \sin 2t.$$

Differentiating again, we get

$$x''(t) = -2(a + 2bt) \sin 2t + 2b \cos 2t + 2(b - 2at) \cos 2t - 2a \sin 2t) = (4b - 4at) \cos 2t - (4a + 4bt) \sin 2t.$$

Then

$$x''(t) + 4x(t) = (4b - 4at) \cos 2t - (4a + 4bt) \sin 2t + 4(at \cos 2t + bt \sin 2t).$$

It follows that

$$4b \cos 2t - 4a \sin 2t = 8 \sin 2t,$$

so that $b = 0$ and $a = -2$. The solution to the nonhomogeneous problem is then

$$x(t) = -2t \cos 2t + c_1 \cos 2t + c_2 \sin 2t.$$

from $x(0) = 0$, we get $c_1 = 0$. Since

$$x'(t) = -4t \sin 2t - 2 \cos 2t + 2c_2 \cos 2t,$$

and $x'(0) = 0$, we get $c_2 = 1$. Therefore, the particular solution is

$$x(t) = -2t \cos 2t + \sin 2t.$$

Resonance occurs when $\omega = 2$ or $\omega = -2$.