92.530 Applied Mathematics I: Solutions to Homework Problems in Chapter 4

- **4.48** Most of the problems here are not difficult. For (g), note that

$$8\sin^2 3t = 4(1 - \cos 6t),$$

which has for its Laplace transform

$$4[\frac{1}{s} - \frac{s}{s^2+36}] = \frac{144}{s(s^2+36)}.$$

- **4.49** (a): From the definition, we have

$$(\mathcal{L}f)(s) = \int_0^4 (-1)e^{-st}dt + \int_4^\infty e^{-st}dt$$

$$= \frac{1}{s}e^{-st}|_0^4 - \frac{1}{s}e^{-st}|_4^\infty$$

$$= \frac{1}{s}(e^{-4s} - 1) - \frac{1}{s}(0 - e^{-4s}) = \frac{1}{s}(2e^{-4s} - 1).$$

- **4.53** The trickiest one is (j): Use partial fractions to write

$$\frac{s}{(s+3)(s+5)} = \frac{-3/2}{s+3} + \frac{5/2}{s+5}.$$

The Laplace transform of e^{-3t} is $\frac{1}{s+3}$, for $s > -3$, and the Laplace transform of e^{-5t} is $\frac{1}{s+5}$, for $s > -5$. Therefore, the answer is $\frac{-3}{2}e^{-3t} + \frac{5}{2}e^{-5t}$.

- **4.55** (a): $f'(t) = 6e^{2t}$, with Laplace transform $\frac{6}{s-2}$. The Laplace transform of $f(t) = 3e^{2t}$ is $\frac{3}{s-2}$, so

$$s(\mathcal{L}f)(s) - f(0) = s(\frac{3}{s-2}) - 3 = \frac{3s}{s-2} - 3 = \frac{3s - 3(s-2)}{s-2} = \frac{6}{s-2} = (\mathcal{L}f')(s).$$

Please Note: The notation in the text is sloppy in some places. For example, the Laplace transform of a function $f(t)$ is a function of the variable s, and we should write $\mathcal{L}\{f(t)\}(s)$, or better, $(\mathcal{L}f)(s)$, or even $\mathcal{L}f(s)$, since the variable t

disappears once the integral in the Laplace transform is performed. In Problem 4.55 the author has written

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(s)\} - f(0),$$

omitting the variable s on the left and putting in the variable s incorrectly in $f(s)$ on the right.

Also note that the table on page 98 has the Laplace transforms of the hyperbolic sine and cosine switched.

- **4.57** Showing that $f(t) = t \sin at$ satisfies the differential equation is easy. Taking the Laplace transform of both sides of

$$f''(t) + a^2 f(t) = 2a \cos at,$$

and using

$$(\mathcal{L}f'')(s) = s^2(\mathcal{L}f)(s) - sf(0) - f'(0),$$

we get

$$s^2(\mathcal{L}f)(s) - sf(0) - f'(0) + a^2(\mathcal{L}f)(s) = 2a\frac{s}{s^2 + a^2}$$

or

$$(s^2 + a^2)(\mathcal{L}f)(s) = 2a\frac{s}{s^2 + a^2} + sf(0) + f'(0) = 2a\frac{s}{s^2 + a^2}.$$

Solving for $(\mathcal{L}f)(s)$, we find that

$$(\mathcal{L}f)(s) = 2a\frac{s}{(s^2 + a^2)^2}.$$

- **4.60** (a): The Laplace transform of $\mathcal{U}(t-1)$ is

$$F(s) = \int_1^\infty e^{-st}dt = \frac{-1}{s}e^{-st}|_1^\infty$$
$$= \frac{-1}{s}[0 - e^{-s}] = \frac{e^{-s}}{s}.$$

Similarly, the Laplace transform of $\mathcal{U}(t-2)$ is $\frac{e^{-2s}}{s}$. Therefore, the answer is

$$\frac{2e^{-s} + 3e^{-2s}}{s}.$$

(b): The Laplace transform of the function $\mathcal{U}(t-3)$ is $\frac{e^{-3s}}{s}$. Multiplying by t corresponds to taking the negative of the derivative with respect to s, so the answer is

$$-\frac{s(-3e^{-3s}) - e^{-3s}}{s^2} = \frac{(3s+1)e^{-3s}}{s^2}.$$

- **4.63** (a): We have

$$f(t) = 3[\mathcal{U}(t) - \mathcal{U}(t - \frac{\pi}{2})] + (\cos t)\mathcal{U}(t - \frac{\pi}{2})$$

$$= 3 + \mathcal{U}(t - \frac{\pi}{2})((\cos t) - 3).$$

(b): Notice that we can write

$$\mathcal{U}(t - \frac{\pi}{2})(\cos t) = \mathcal{U}(t - \frac{\pi}{2})(-\sin(t - \frac{\pi}{2})).$$

Applying Theorem 4-6, we find that the Laplace transform of $\mathcal{U}(t - \frac{\pi}{2})(\cos t)$ is

$$-\frac{e^{-s\pi/2}}{s^2 + 1}.$$

The answer, therefore, is

$$\frac{3}{s} - \frac{3e^{-s\pi/2}}{s} - \frac{e^{-s\pi/2}}{s^2 + 1},$$

or

$$\frac{3}{s}(1 - e^{\frac{-s\pi}{2}}) - \frac{e^{-s\pi/2}}{s^2 + 1}.$$

- **4.67** (b): We use Theorem 4-9. The period is 4 and, from integration by parts, the integral is

$$\int_0^4 e^{-st} t dt = \frac{1}{s^2}(1 - e^{-4s}) - \frac{4}{s}e^{-4s}.$$

Dividing by $1 - e^{-4s}$, we get the answer,

$$\frac{1 - e^{-4s} - 4se^{-4s}}{s^2(1 - e^{-4s})}.$$

- **4.71** (b): We use Theorem 4-11. The Laplace transform of the numerator $\cos 2t - \cos 3t$ is

$$\frac{s}{s^2 + 4} - \frac{s}{s^2 + 9}.$$

Dividing by t corresponds to integrating the Laplace transform. We compute

$$\int_s^\infty \frac{u}{u^2 + 4} - \frac{u}{u^2 + 9} du,$$

obtaining

$$\frac{1}{2}[\log(u^2 + 4) - \log(u^2 + 9)] \,|_s^\infty.$$

The answer is therefore

$$\frac{1}{2}[\log \frac{u^2 + 4}{u^2 + 9}] \,|_s^\infty = \frac{1}{2}[\log(1) - \log \frac{s^2 + 4}{s^2 + 9}] = \log \sqrt{\frac{s^2 + 9}{s^2 + 4}}.$$

(c): The Laplace transform of $\sin t$ is $\frac{1}{s^2+1}$. Dividing by t corresponds to integrating from s to ∞; that is, the Laplace transform of $\frac{\sin t}{t}$ is

$$\int_s^{+\infty} \frac{1}{u^2+1} du = \tan^{-1}(+\infty) - \tan^{-1}(s).$$

Since $\tan^{-1}(+\infty) = \frac{\pi}{2}$, we can write the answer as

$$\frac{\pi}{2} - \tan^{-1}(s) = \cot^{-1}(s) = \tan^{-1}(\frac{1}{s}).$$

- **4.79** We write

$$\log(s+6) - \log(s+2) = \int_s^{\infty} \frac{1}{u+2} - \frac{1}{u+6} du = [\log \frac{u+2}{u+6}] |_s^{\infty}.$$

The inverse Laplace transform of $\frac{1}{u+2} - \frac{1}{u+6}$ is $e^{-2t} - e^{-6t}$, so, using Theorem 4-11, we get the final answer,

$$\frac{e^{-2t} - e^{-6t}}{t}.$$

- 4.82 This one is not easy; I suggest ignoring the hint. Substitute $u = x^2$, so that $dx = \frac{du}{2\sqrt{u}}$. The integral then is

$$\int_0^{\infty} (\sin u) \frac{1}{2\sqrt{u}} du.$$

The trick is to write

$$\frac{1}{\sqrt{u}} = \frac{1}{\Gamma(\frac{1}{2})} \int_0^{\infty} \frac{1}{\sqrt{t}} e^{-ut} dt.$$

Inserting this into the previous integral and switching the order of integration, we get

$$\int_0^{\infty} \sin x^2 dx = \int_0^{\infty} \frac{1}{\sqrt{t}} \int_0^{\infty} \sin u e^{-ut} du] dt.$$

The inner integral is the Laplace transform of $\sin u$, so we have

$$\int_0^{\infty} \sin x^2 dx = \int_0^{\infty} \frac{1}{\sqrt{t}} \frac{1}{t^2+1} dt.$$

Letting $r^2 = t$, so that $dt = 2rdr$, we have

$$\int_0^{\infty} \sin x^2 dx = \frac{1}{\Gamma(\frac{1}{2})} \int_0^{\infty} \frac{1}{r^4+1} dr.$$

This last integral can be done using Problem 13.29, or we can look up its value, which turns out to be $\frac{\pi\sqrt{2}}{4}$. The answer then is $\frac{1}{2}\sqrt{\frac{\pi}{2}}$.

- **4.84** This problem is not difficult, but it does require a lot of patience and careful calculation. It is the sort of problem you must do once in your life, but probably not twice. Applying Theorem 4-3 to the differential equation, we get

$$(s^3 + 8)Y(s) = s^{-4}(192 - 16s^2),$$

where $Y(s)$ is the Laplace transform of the function $y(t)$ that we seek. Solving for $Y(s)$, we have

$$Y(s) = \frac{192 - 16s^2}{s^4(s+2)(s^2 - 2s + 4)},$$

where we make use of the factorization $s^3 + 8 = (s+2)(s^2 - 2s + 4)$. The next step is to rewrite the rational expression using partial fractions. We set

$$\frac{192 - 16s^2}{s^4(s+2)(s^2 - 2s + 4)} = \frac{A}{s^4} + \frac{B}{s^3} + \frac{C}{s^2} + \frac{D}{s} + \frac{E}{s+2} + \frac{Fs + G}{s^2 - 2s + 4}.$$

Taking a common denominator again, and equating the numerators on both sides, we eventually find that $A = 24$, $B = 0$, $C = -2$, $D = -3$, $E = 2/3$, $F = 7/3$, and $G = -4/3$. Inserting these values, we find that

$$Y(s) = \frac{24}{s^4} - \frac{2}{s^2} - \frac{3}{s} + \frac{2}{3}\frac{1}{s+2} + \frac{1}{3}\frac{7s - 4}{s^2 - 2s + 4}.$$

Taking the inverse Laplace transform of each term is easy, with the possible exception of the last term. We simplify the last term by writing

$$\frac{7s - 4}{s^2 - 2s + 4} = \frac{7(s-1) + 3}{(s-1)^2 + 3},$$

which has the inverse Laplace transform

$$7e^t \cos(\sqrt{3}t) + \sqrt{3}e^t \sin(\sqrt{3}t).$$

- **4.85** Like the previous problem, this one is not really difficult, but does require a bit of patience (and maybe a bit of luck!) to get it right the first time; most of it is *just algebra*, but there are lots of opportunities to make dumb mistakes.

  Note that the initial conditions $x(0) = 0$ and $y(0) = 0$, along with the differential equations themselves, imply that $x'(0) = 4$ and $y'(0) = 0$. Solving for y in the second equation, then differentiating, then substituting into the first equation gives us a second-order equation in $x(t)$ only:

$$x'' + x' + x = 3 - 2e^{-t} - 3e^{-2t}.$$

Taking the Laplace transform of both sides and using Theorem 4-3, we get

$$X(s)(s^2+s+1)-4=\frac{3}{s}-\frac{2}{s+1}-\frac{3}{s+2},$$

or

$$X(s)(s^2+s+1)=4+\frac{3}{s}-\frac{2}{s+1}-\frac{3}{s+2}=\frac{4s^3+10s^2+10s+6}{s(s+1)(s+2)}.$$

Therefore, we have

$$X(s)=\frac{4s^3+10s^2+10s+6}{s(s+1)(s+2)(s^2+s+1)}.$$

Now we write

$$X(s)=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}+\frac{Ds+E}{s^2+s+1}.$$

Taking the common denominator and equating numerators, we eventually find that $A=3$, $B=-2$, $C=-1$, and $D=E=0$. Therefore,

$$X(s)=\frac{3}{s}-\frac{2}{s+1}-\frac{1}{s+2},$$

so that

$$x(t)=3-2e^{-t}-e^{-2t}.$$

From this we get

$$x'(t)=2e^{-t}+2e^{-2t},$$

so that

$$y(t)=2+6e^{-2t}-2x'(t)=2+6e^{-2t}-4e^{-t}-4e^{-2t}=2+2e^{-2t}-4e^{-t}.$$

- **4.91** The *Dirac delta* $\delta(t)$ is not really a function, but a useful *generalized function*. Most of the time, we deal with the Dirac delta in a mechanical, formal way, without asking too many questions. For example, once we agree that, for (almost) any function $f(t)$,

$$\int_0^\infty f(t)\delta(t)dt=f(0),$$

then it follows that

$$\int_0^\infty f(t)\delta(t-t_0)dt=f(t_0).$$

To see this, we substitute $u=t-t_0$, so that

$$\int_0^\infty f(t)\delta(t-t_0)dt=\int_{-t_0}^\infty f(u+t_0)\delta(u)du=f(0+t_0)=f(t_0).$$

(a): Now we compute $\Delta(s)$, the Laplace transform of $\delta(t)$:

$$\Delta(s) = \int_0^\infty e^{-st}\delta(t)dt = e^{-0} = 1.$$

Sometimes the Dirac delta is even defined as the "function" whose Laplace transform is the function that equals one for all s.

(c): The book hasn't really given enough information to do this one rigorously. The first step would be to extend the theorem on the Laplace transform of the derivative, Theorem 4.3 on page 100, to functions $f(t)$ where $\lim_{\epsilon \downarrow 0} f(t) = f(0+)$ exists, but may not equal $f(0)$. The theorem would then read

$$\mathcal{L}(f'(t))(s) = s\mathcal{L}(f(t))(s) - f(0+).$$

Although the delta function is not truly a function, we may think of it as being zero for any t other than $t = 0$, so that $\delta(0+) = 0$. Since, by the extended theorem, differentiating $\delta(t)$ corresponds to multiplying the Laplace transform of $\delta(t)$ by s, it follows that the Laplace transform of the derivative of the Dirac delta must be the function that is s everywhere. Of course, since the Dirac delta is not really a function, it doesn't really have a derivative, at least in the ordinary sense.

Since the Laplace transform of the function $\mathcal{U}(t)$ is $\frac{1}{s}$, it follows that the derivative of $\mathcal{U}(t)$ would have for its Laplace transform s times $\frac{1}{s}$, that is, the derivative of $\mathcal{U}(t)$ would be a function whose Laplace transform is one for all values of s, that is, the derivative is the Dirac delta. Of course, the function $\mathcal{U}(t)$ is discontinuous at $t = 0$, so does not possess a derivative in the ordinary sense. It is fitting, then, that its *derivative* is not a function, but something more general.

An ordinary function $g(t)$ can be used to create a *functional*, that is, a real-valued *function of functions*: we simply calculate, for every suitable $f(t)$, the integral

$$\int_0^\infty f(t)g(t)dt.$$

Another functional is the one that associates with every $f(t)$ its value at $t = t_0$. This functional does not come from a function $g(t)$ as above, but if we write it as though it did, we use the Dirac delta; that is,

$$f(t_0) = \int_0^\infty f(t)\delta(t - t_0)dt.$$

The Dirac delta is a perfectly ordinary object when thought of as just a functional, but becomes mysterious when we try to view it as coming from a function $g(t)$.

All this may sound a bit crazy, but it can be made mathematically sound. What we are talking about here is *distribution theory*, in which integration by parts plays a key role.