92.531 Applied Mathematics II: Solutions to Homework Problems in Chapter 5

- **5.46** The text does not actually give a definition of *the plane determined by two vectors*. Clearly here the two vectors are to be considered as directed line segments. If we situate the two vectors so that they have a common initial point, then they determine a plane containing that initial point. The problem then asks us to show that, if we form the directed line segment $mA + nB$ and have its initial point coincide with that of A and B, then this directed line segment will lie in the plane previously determined by A and B.

  In worked problems 5.23 and 5.24, the author uses in the solutions the fact that a vector C lies in the plane of two other non-parallel vectors A and B if and only if the triple product $C \cdot (A \times B) = 0$. So we can use that idea here. We show that
$$(mA + nB) \cdot (A \times B) = 0.$$
  We know that
$$(mA + nB) \cdot (A \times B) = mA \cdot (A \times B) + nB \cdot (A \times B).$$
  Also, by Problem 5.22,
$$A \cdot (A \times B) = (A \times A) \cdot B = 0,$$
  and
$$B \cdot (A \times B) = -B \cdot (B \times A) = -(B \times B) \cdot A = 0.$$
  Consequently, the vector $mA + nB$ lies in the plane determined by A and B, for every real numbers m and n.

- **5.49** Let two of the sides of the triangle be formed by placing vectors A and B with their initial ends at the origin. The vectors A and B form two of the three sides, and the vector $A - B$ forms the third side. Let $C = \frac{1}{2}(A + B)$ be the vector from the origin to the midpoint of the opposite side, and $D = A - \frac{1}{2}B$

be the vector from the midpoint of vector B to the terminal end of vector A. A picture might help you here. Then vectors C and D are median line segments for the triangle and meet at the point in question. Writing

$$\beta C = \frac{1}{2}B + \alpha D,$$

and substituting, we find that

$$\frac{1}{2}(1-(\alpha+\beta))B = (\frac{\beta}{2}-\alpha)A.$$

Since the vectors A and B are not parallel, we must have

$$1-(\alpha+\beta) = 0,$$

and

$$\beta - 2\alpha = 0,$$

so that $\beta = \frac{2}{3}$, and $\alpha = \frac{1}{3}$. This proves the assertion.

- **5.59** The cross product is $A \times B = (0, 10, 5)$, so a vector perpendicular to both A and B is $(0, 2, 1)$, having length $\sqrt{5}$. Therefore, $\frac{1}{\sqrt{5}}(0, 2, 1)$ is an answer.

- **5.61** With the notation $P_1 = (1, -1, 2)$, $P_2 = (2, -3, 1)$, and $P_3 = (-1, 2, 3)$, let A be the vector $P_2P_1 = (1, -2, -1)$ from P_1 to P_2, and $B = P_3P_1 = (-2, 3, 1)$. The cross product is then $A \times B = (1, 1, -1)$, with length $\sqrt{3}$, which is twice the area of the triangle.

- **5.62** Let $P_1 = (1, 1, 0)$, $P_2 = (3, -1, 1)$, and $P_3 = (-1, 0, 2)$. The vectors $A = P_2P_1 = (2, -2, 1)$ and $B = P_3P_1 = (-2, -1, 2)$ lie in the plane. A normal vector is $A \times B = (-3, -6, -6)$, so that $(1, 2, 2)$ is also a normal vector. The equation of the plane is then

$$x + 2y + 2z = 3.$$

The nearest point is $P = (x, y, z)$ with the vector $(3-x, 2-y, 1-z)$ parallel to the normal vector $(1, 2, 2)$ and the point $P = (x, y, z)$ in the plane. Therefore, $x = 3 + c$, $y = 2 + 2c$, and $z = 1 + 2c$. For (x, y, z) to be on the plane, we need $c = -2/3$. The nearest point is then $(7/3, 2/3, -1/3)$ and the distance is 2.

- **5.65** See problem 5.24.

- **5.68** The velocity vector is

$$r' = (-e^{-t})(\sin t + \cos t, \sin t - \cos t, 1),$$

and the acceleration vector is

$$r'' = (e^{-t})(2\sin t, -2\cos t, 1).$$

The magnitude of r' is

$$|r'| = \sqrt{3}e^{-t},$$

and the magnitude of r'' is

$$|r''| = \sqrt{5}e^{-t}.$$

- **5.69** Use the notation $A(u) = (A_1(u), A_2(u), A_3(u))$ and $B(u) = (B_1(u), B_2(u), B_3(u))$. Then form $A(u) \times B(u)$ and differentiate.

- **5.74** Use the notation $A(u) = (A_1(u), A_2(u), A_3(u))$, so that

$$1 = |A(u)|^2 = A_1(u)^2 + A_2(u)^2 + A_3(u)^2.$$

Differentiating, we get

$$0 = 2A_1(u)A_1'(u) + 2A_2(u)A_2'(u) + 2A_3(u)A_3'(u).$$

Therefore, $A(u) \cdot A'(u) = 0$.

- **5.77** Writing

$$A(x,y,z) = (A_1(x,y,z), A_2(x,y,z), A_3(x,y,z)),$$

with

$$A_1(x,y,z) = x^2y,$$

$$A_2(x,y,z) = y^2z,$$

and

$$A_3(x,y,z) = z^2x,$$

we have

$$\nabla \cdot A = \frac{\partial}{\partial x}A_1 + \frac{\partial}{\partial y}A_2 + \frac{\partial}{\partial z}A_3 = 2(xy + yz + xz)$$

so that, at the point $(3,-1,2)$, we have

$$A(3,-1,2) = (-9, 2, 12),$$

$$\nabla\phi(3,-1,2) = (1,5,2),$$

and

$$(\nabla \cdot A)(3,-1,2) = 2.$$

Therefore,

$$A \cdot \nabla\phi = 25,$$

$$\phi\nabla \cdot A = (\phi(3,-1,2))((\nabla \cdot A)(3,-1,2)) = (1)(2) = 2,$$

and

$$(\nabla\phi \times A)(3,-1,2) = (56,-30,47).$$

- **5.78** With $\mathbf{r}(x,y,z) = (x,y,z)$, we have

$$r^2 = |\mathbf{r}|^2 = x^2 + y^2 + z^2,$$

and

$$r^2\mathbf{r} = (x(x^2+y^2+z^2), y(x^2+y^2+z^2), z(x^2+y^2+z^2)) = (a,b,c).$$

Then, using this shorthand, we have

$$\text{curl } = (c_y - b_z, a_z - c_x, b_x - a_y).$$

The first term is

$$c_y - b_z = 2yz - 2yz = 0,$$

and the other two terms are zero also.

- **5.82** We have

$$\operatorname{curl} A = (y, 6xz-1, 0),$$

so that

$$\operatorname{curl}\operatorname{curl} A = \operatorname{curl}(y, 6xz-1, 0) = (-6x, 0, 6z-1).$$

- **5.83** Use the shorthand $A = (a,b,c)$, where a, b, and c are functions of (x,y,z). Note that, by definition,

$$\nabla^2 A = (\nabla^2 a, \nabla^2 b, \nabla^2 c).$$

The first component of the vector $\nabla \times (\nabla \times A)$ is

$$b_{xy} - a_{yy} - a_{zz} + c_{xz}.$$

The first component of the vector $\nabla(\nabla \cdot A)$ is

$$a_{xx} + b_{yx} + c_{zx},$$

and the first component of the vector $-\nabla^2 A$ is

$$-a_{xx} - a_{yy} - a_{zz}.$$

The identity is clearly true for the first component; the other two are done in the same way.

- **5.89** Note that the equations in (1) are Maxwell's equations in any region of space where there are no currents or charges. The first two equations are Gauss's Law for the electric and magnetic fields, respectively, the third equation is Faraday's Law, and the fourth one is the Maxwell-Ampere Law.

  We use the identity from Problem 5.83,

  $$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A,$$

  first with $A = H$, and then with $A = E$; we do only $A = H$ here. So, we have

  $$\nabla \times (\nabla \times H) = \nabla(\nabla \cdot H) - \nabla^2 H = -\nabla^2 H,$$

  since $\nabla \cdot H = 0$, and

  $$\nabla \times (\nabla \times H) = \nabla \times (\frac{1}{c}\frac{\partial E}{\partial t}) = \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times E) = \frac{1}{c}\frac{\partial}{\partial t}(\frac{-1}{c}\frac{\partial H}{\partial t}) = -\frac{1}{c^2}\frac{\partial^2 H}{\partial t^2}.$$

  So

  $$\nabla^2 H = \frac{1}{c^2}\frac{\partial^2 H}{\partial t^2}.$$

- **5.90** For clarity now, we use $\mathbf{E}$ and $\mathbf{H}$ for the fields, $E^2 = \mathbf{E} \cdot \mathbf{E}$ and similarly for H^2. Use the identity

  $$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}),$$

  and

  $$\mathbf{H}\frac{\partial \mathbf{H}}{\partial t} + \mathbf{E}\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}(\frac{1}{2}(\mathbf{H} \cdot \mathbf{H} + \mathbf{E} \cdot \mathbf{E})) = \frac{\partial}{\partial t}(\frac{1}{2}(H^2 + E^2)).$$