92.531 Applied Mathematics II: Solutions to Homework Problems in Chapter 6

- **6.44** View the volume as having as the floor the two-dimensional region bounded by $\frac{x}{a} + \frac{y}{b} = 1$ and the x and y axes, with the roof given by

$$z = c(1 - \frac{x}{a} - \frac{y}{b}).$$

The integral is then

$$\int_{x=0}^{x=a} \int_{y=0}^{y=b(1-\frac{x}{a})} c(1 - \frac{x}{a} + \frac{y}{b}) dy\, dx.$$

The inner integral is

$$\int_{y=0}^{y=b(1-\frac{x}{a})} c(1 - \frac{x}{a} - \frac{y}{b}) dy = c(1 - \frac{x}{a})y - \frac{c}{2b} y^2 \mid_{y=0}^{y=b(1-\frac{x}{a})}$$

$$= c(1 - \frac{x}{a}) b(1 - \frac{x}{a}) - \frac{c}{2b}(b(1 - \frac{x}{a}))^2$$

$$= \frac{1}{2} cb(1 - \frac{x}{a})^2.$$

The outer integral is then

$$\frac{1}{2} cb \int_0^a (1 - \frac{2x}{a} + \frac{1}{a^2} x^2) dx = \frac{abc}{6}.$$

- **6.45** The integral is

$$\int_{x=-a}^{x=a} \int_{y=-a}^{y=a} x^2 + y^2 dy\, dx.$$

The inner integral is

$$\int_{y=-a}^{y=a} x^2 + y^2 dy = x^2 y + \frac{1}{3} y^3 \mid_{-a}^{a} = 2ax^2 + \frac{2}{3} a^3,$$

so the outer integral is

$$\int_{-a}^{a} 2ax^2 + \frac{2}{3} a^3 dx = \frac{2a}{3} x^3 \mid_{-a}^{a} + \frac{2a^3}{3} x \mid_{-a}^{a} = \frac{2a}{3}(2a^3) + \frac{4a^4}{3} = \frac{8a^4}{3}.$$

- **6.50** The volume has, for its floor, the quarter of the circle $x^2+y^2=4$ that lies within the first quadrant, and its roof is

$$z=\sqrt{4-x^2-y^2}.$$

So the integral is

$$\int_{x=0}^{x=2}\int_0^{y=\sqrt{4-x^2}}\int_{z=0}^{z=\sqrt{4-x^2-y^2}} xyzdz\,dy\,dx.$$

The inner integral is

$$\frac{xy}{2}z^2\,|_{z=0}^{z=\sqrt{4-x^2-y^2}}=\frac{xy}{2}(4-x^2-y^2).$$

The second integral is then

$$\int_0^{y=\sqrt{4-x^2}}\frac{xy}{2}(4-x^2-y^2)dy=\frac{4x-x^3}{4}y^2-\frac{x}{8}y^4\,|_{y=0}^{y=\sqrt{4-x^2}}$$
$$=\frac{4x-x^3}{4}(4-x^2)-\frac{x}{8}(4-x^2)^2=(\frac{x}{8})(4-x^2)^2.$$

The outer integral is then

$$\int_0^2(\frac{x}{8})(4-x^2)^2dx=\int_0^2 2x-x^3+\frac{1}{8}x^5dx=x^2-\frac{1}{4}x^4+\frac{1}{48}x^6\,|_0^2=4-4+\frac{64}{48}=\frac{4}{3}.$$

- **6.51** The volume lies above the circle in the x,y-plane with center at $(x,y)=(1,0)$ and radius one. The floor is $z=x^2+y^2$ and the roof is $z=2x$. The integral is then

$$\int_{x=0}^{x=2}\int_{y=-\sqrt{2x-x^2}}^{y=\sqrt{2x-x^2}}2x-x^2-y^2dy\,dx.$$

The inner integral is

$$\int_{y=-\sqrt{2x-x^2}}^{y=\sqrt{2x-x^2}}2x-x^2-y^2dy=2xy-x^2y-\frac{1}{3}y^3\,|_{y=-\sqrt{2x-x^2}}^{y=\sqrt{2x-x^2}}$$
$$=4x\sqrt{2x-x^2}-2x^2\sqrt{2x-x^2}-\frac{2}{3}(2x-x^2)^{3/2}=\frac{1}{3}(2x-x^2)\sqrt{2x-x^2}.$$

With the substitution of $t=x-1$ and $\cos\theta=\sqrt{1-t^2}$, the outer integral becomes

$$\frac{8}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta=\frac{\pi}{2}.$$

- **6.67** We do part (a). The parabola $y^2=x$ can be parameterized as (t^2,t), with $t=1$ to $t=2$, and $\frac{dx}{dt}=2t$, and $\frac{dy}{dt}=1$. Then

$$\int_{(1,1)}^{(4,2)}(x+y)dx+(y-x)dy=\int_{t=1}^{t=2}\Big((t^2+t)(2t)+(t-t^2)(1)\Big)dt=\int_1^2 2t^3+t^2+tdt=\frac{34}{3}.$$

- **6.68** We can describe the base of the triangle as $(t, 0)$, for $0 \leq t \leq 3$, so $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 0$, and the line integral along the base is

$$\int_0^3 2t + 4dt = 21.$$

The vertical side is $(3, t)$, for $0 \leq t \leq 2$, with $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 1$. The line integral along the vertical side is then

$$\int_0^2 5t + 9 - 6dt = \int_0^2 5t + 3dt = 16.$$

We can let the hypotenuse be described by $(3 - 3t, 2 - 2t)$, for $0 \leq t \leq 1$, so that $\frac{dx}{dt} = -3$ and $\frac{dy}{dt} = -2$. The line integral down the hypotenuse is then

$$\int_0^1 12t - 24 + 38t - 26dt = -25.$$

So the line integral around the perimeter of the triangle is $21 + 16 - 25 = 12$.

- **6.75** The line integral in problem 6.68 is

$$\oint (2x - y + 4)dx + (5y + 3x - 6)dy.$$

With $P(x, y) = 2x - y + 4$, $Q(x, y) = 5y + 3x - 6$, $Q_x(x, y) = 3$, and $P_y(x, y) = -1$, Green's Theorem in the plane says that

$$\oint (2x - y + 4)dx + (5y + 3x - 6)dy = \int\int (Q_x - P_y)dx\, dy = \int\int (4)dx\, dy.$$

Therefore, the line integral is four times the area of the triangle, that is, four times three, or twelve. That was easier, wasn't it?

- **6.84** The equation of the cone is

$$z = \sqrt{3}\sqrt{x^2 + y^2}.$$

Then

$$z_x = \sqrt{3}\frac{1}{2}\frac{2x}{\sqrt{x^2 + y^2}} = \sqrt{3}\frac{x}{\sqrt{x^2 + y^2}},$$

and

$$z_y = \sqrt{3}\frac{y}{\sqrt{x^2 + y^2}}.$$

Then

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + 3} = 2.$$

Following Problem 6.25, the surface integral reduces to the double integral

$$\int\int_{\mathcal{R}} (x^2+y^2)(2)dx\,dy,$$

where $\mathcal{R}$ is the circle of radius $\sqrt{3}$, centered at the origin. The integral is best done in polar coordinates, as

$$\int\int_{\mathcal{R}} (x^2+y^2)(2)dx\,dy, = 2\int_0^{2\pi}\int_0^{\sqrt{3}} r^2 r\,dr\,d\theta$$

$$= 2\int_0^{2\pi} \frac{1}{4}r^4\,|_0^{\sqrt{3}} d\theta = 2\int_0^{2\pi}\frac{1}{4}(\sqrt{3})^4 d\theta = \frac{9}{2}\int_0^{2\pi} d\theta = \frac{9}{2}(2\pi) = 9\pi.$$

The physical interpretation of the surface integral is that it represents the moment of inertia of the cone, with respect to revolution about the z-axis.

- **6.92** With $A(x,y,z)$ the field

$$A(x,y,z) = (2xy+z, y^2, -x-3y),$$

the divergence of A is

$$\text{div}\, A = \nabla \cdot A = 2y + 2y = 4y.$$

We consider first the triple integral

$$\int\int\int_V \nabla\cdot A dV = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x}\int_{z=0}^{z=6-2x-2y} 4ydz\,dy\,dx.$$

Therefore,

$$\int\int\int_V \nabla\cdot A dV = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} 4y(6-2x-2y)dy\,dx$$

$$= \int_{x=0}^{x=3}\int_{y=0}^{y=3-x}(24y-8xy-8y^2)dy\,dx = \int_{x=0}^{x=3}(12y^2-4xy^2-\frac{8}{3}y^3)\,|_{y=0}^{y=3-x}\,dx$$

$$= \int_{x=0}^{x=3}(12(3-x)^2-4x(3-x)^2-\frac{8}{3}(3-x)^3)\,dx$$

$$= \int_{x=0}^{x=3}(108-72x+12x^2-36x+24x^2-4x^3-72+72x-24x^2+\frac{8}{3}x^3)\,dx$$

$$\int_{x=0}^{x=3}(36-36x+12x^2-\frac{4}{3}x^3)\,dx = 36(3)-18(9)+108-27 = 216-162-27 = 216-189 = 27.$$

Now we consider the surface integral. The outward normal to the surface is the constant vector

$$n = (2,2,1),$$

so

$$\int\int_S A \cdot ndS = \int\int_S 2(2xy+z) + 2(y^2) + (-x-3y)dS.$$

With

$$z = 6 - 2x - 2y,$$

we have

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1+4+4} = 3.$$

Therefore,

$$\int\int_S A \cdot ndS = 3\int_{x=0}^{x=3}\int_{y=0}^{y=3-x} 2(2xy+z) + 2(y^2) + (-x-3y)dy\,dx$$

$$= 3\int_{x=0}^{x=3}\int_{y=0}^{y=3-x}(4xy + 2(6-2x-2y) + 2y^2 - x - 3y)dy\,dx$$

$$= 3\int_{x=0}^{x=3}\int_{y=0}^{y=3-x}(12 - 5x - 7y + 4xy + 2y^2)dy\,dx$$

$$= 3\int_{x=0}^{x=3}(12(3-x) - 5x(3-x) - \frac{7}{2}(3-x)^2 + 2x(3-x)^2 + \frac{2}{3}(3-x)^3)\,dx$$

$$= \int_0^3(\frac{45}{2} - \frac{9}{2}x^2 + \frac{4}{3}x^3)dx = 54.$$

- **6.93** The field is $F(x,y,z) = (z^2 - x, -xy, 3z)$, the surface is the tunnel-shaped object bounded by $z = 4 - y^2$, $x = 0$, $x = 3$, and with the floor the x, y-plane. If you try calculating the surface integral directly, you quickly find that it is a mess. Instead, let's use the Divergence Theorem. The divergence of the field F is the function $2 - x$, which we then have to integrate over the volume. The integral is then

  $$\int_{x=0}^{x=3}\int_{y=-2}^{y=2}\int_{z=0}^{z=4-y^2}(2-x)dz\,dy\,dx.$$

  The integration is easy and the answer is 16.

- **6.97** In this problem, as in the previous one, we are asked to calculate a surface integral of the form $\int\int_S F \cdot ndS$; now the field F is $F = \nabla \times A$, for some field A. As in the previous problem, the Divergence Theorem is the best way to proceed. The divergence of the field $F = \nabla \times A$ is

  $$\nabla \cdot F = \nabla \cdot (\nabla \times A) = \operatorname{div}\operatorname{curl} A,$$

  which equals zero, by (11) on page 127.

- **6.100** Note that the curve C is the boundary of the circular region $\mathcal{R}$ that is capped by the surface.

  We first consider the double integral. The curl of A is $(0,0,1)$ and the unit normal n is

  $$n = \frac{1}{6}(2x, 2y, 2\sqrt{9 - x^2 - y^2}).$$

  Also

  $$\sqrt{1 + z_x^2 + z_y^2} = 3/\sqrt{9 - x^2 - y^2}.$$

  The double integral is then

  $$\int\int_S (\nabla \times A) \cdot n dS = \int\int_{\mathcal{R}} dy\, dx,$$

  which is the area of the circular region $\mathcal{R}$ enclosed by the curve C, so equals 9π.

  Now we consider the line integral. The curve C is the circle centered at the origin, with radius three, so it has the representation $(3\cos\theta, 3\sin\theta)$, for $0 \leq \theta \leq 2\pi$. The tangent vector to C is $t = (-3\sin\theta, 3\cos\theta, 0)$, so

  $$A \cdot t = (2y)(-3\sin\theta) + (3x)(3\cos\theta) = -18\sin^2\theta + 27\cos^2\theta = 18\cos(2\theta) + 9\cos^2\theta.$$

  The line integral is then

  $$18\int_{\theta=0}^{\theta=2\pi} \cos(2\theta) d\theta + 9\int_{\theta=0}^{\theta=2\pi} \cos^2\theta\, d\theta.$$

  The first integral is zero, and

  $$9\int_{\theta=0}^{\theta=2\pi} \cos^2\theta\, d\theta = 9\int_{\theta=0}^{\theta=2\pi} (\frac{1}{2} + \frac{1}{2}\cos(2\theta))\, d\theta.$$

  The integral of the cosine term is zero, while the integral of the constant term gives us $(9)(\frac{1}{2})(2\pi) = 9\pi$.

- **6.104** In order for a field

  $$F(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$$

  to be conservative, we need

  $$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x},$$
  $$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x},$$

and

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

The reason for this is that a conservative field is a gradient field, that is, there is a real-valued function $\phi(x, y, z)$ such that $F(x, y, z) = \nabla\phi(x, y, z)$. Therefore, we have

$$F_1(x, y, z) = \frac{\partial \phi}{\partial x},$$

$$F_2(x, y, z) = \frac{\partial \phi}{\partial y},$$

and

$$F_3(x, y, z) = \frac{\partial \phi}{\partial z}.$$

Then

$$\frac{\partial F_1}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial F_2}{\partial x},$$

and similarly for the other two conditions above.

(a) Now we have

$$F(x, y, z) = (2xy + 3, x^2 - 4z, -4y).$$

Then

$$\frac{\partial F_1}{\partial y} = 2x,$$

$$\frac{\partial F_2}{\partial x} = 2x,$$

$$\frac{\partial F_1}{\partial z} = 0,$$

$$\frac{\partial F_3}{\partial x} = 0,$$

$$\frac{\partial F_2}{\partial z} = -4,$$

and

$$\frac{\partial F_3}{\partial y} = -4,$$

so the conditions hold for F to be conservative.

(b) Now we need to find a function $\phi(x, y, z)$ with $F(x, y, z) = \nabla\phi(x, y, z)$. We begin with

$$F_1 = 2xy + 3 = \frac{\partial \phi}{\partial x}.$$

It follows that

$$\phi(x, y, z) = x^2 y + 3x + g(y, z),$$

for some function $g(y,z)$ of y and z only. Then

$$F_2 = x^2 - 4z = \frac{\partial \phi}{\partial y} = x^2 + \frac{\partial g}{\partial y},$$

so

$$\frac{\partial g}{\partial y} = -4z,$$

and

$$g(y,z) = -4yz + h(z),$$

for some function $h(z)$ of z only. So far, then, we have

$$\phi = x^2y + 3x - 4yz + h(z).$$

Therefore,

$$-4y = F_3 = \frac{\partial \phi}{\partial z} = -4y + h'(z),$$

from which we can conclude that $h(z)$ is a constant function, which we can then ignore. So we have found ϕ; it is

$$\phi(x,y,z) = x^2y + 3x - 4yz + \text{constant}.$$

(c) Since the line integral is

$$\int_C F \cdot dr = \int_{t=t_0}^{t=t_1} (F_1, F_2, F_3) \cdot (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})dt,$$

and, by the Chain Rule,

$$\frac{d}{dt}\phi(x(t), y(t), z(t)) = (F_1, F_2, F_3) \cdot (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})dt,$$

it follows that

$$\int_C F \cdot dr = \int_{t=t_0}^{t=t_1} \frac{d\phi}{dt} dt = \phi(x(t_1), y(t_1), z(t_1)) - \phi(x(t_0), y(t_0), z(t_0)),$$

where $t_0 \leq t \leq t_1$ are the limits of the parameter t.

Therefore, for any path C connecting $(3,-1,2)$ and $(2,1,-1)$, the line integral $\int_C F \cdot dr$ is equal to the difference

$$\phi(2,1,-1) - \phi(3,-1,2) = 14 - 8 = 6.$$

We illustrate this point by computing the line integral directly, for a particular parametrization; the reader is welcome to try another. Let the curve C be described by $(3-t,-1+2t,2-3t)$, for $0 \le t \le 1$. Then $\frac{dx}{dt}=-1$, $\frac{dy}{dt}=2$, and $\frac{dz}{dt}=-3$. Therefore,

$$\int_C F\cdot dr = \int_0^1 (2(3-t)(-1+2t)+3,(3-t)^2-4(2-3t),-4(-1+2t))\cdot(-1,2,-3)dt$$

$$= \int_0^1 (6t^2+22t-7)dt = 6.$$