

## 92.530 Applied Mathematics I: Solutions to Homework Problems in Chapter 7

- **7.26(d)** Since the period is given as  $2L = 6$ , we seek a Fourier series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{3}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{3}x\right).$$

Using integration by parts once, we calculate

$$a_n = \frac{1}{3} \int_{-3}^3 f(x) \cos\left(\frac{n\pi}{3}x\right) dx = \frac{1}{3} \int_0^3 2x \cos\left(\frac{n\pi}{3}x\right) dx$$

and find that

$$a_n = 6 \frac{(\cos(n\pi) - 1)}{(n\pi)^2}.$$

Also

$$a_0 = \frac{1}{3} \int_0^3 2x dx = 3.$$

Similarly,

$$b_n = \frac{1}{3} \int_0^{\infty} 2x \sin\left(\frac{n\pi}{3}x\right) dx = -6 \frac{\cos(n\pi)}{n\pi}.$$

- **7.27** In part (a), the discontinuities are at  $x = 2$ , and again at every point having the form  $x = 2 + 4m$ , for any integer  $m$ . The Fourier series converges to 0, the mean of 8 and  $-8$ , at these discontinuity points. In part (b),  $f(x)$  has no discontinuities. In part (c),  $f(x)$  has a discontinuity at  $x = 0$ , and again at every point of the form  $x = 10m$ , for any integer  $m$ . The Fourier series converges, at these points of discontinuity, to the value 20. Finally, in part (d),  $f(x)$  has a discontinuity at  $x = 3$  and again at every point of the form  $x = 3 + 6m$ , for any integer  $m$ . The Fourier series converges, at these points of discontinuity, to the value 3.
- **7.29** When we extend  $f(x)$  periodically, with period  $2L = \pi$ , we get an odd function. Therefore, its Fourier series is automatically a sine series,

$$\sum_{n=1}^{\infty} b_n \sin(2nx),$$

with

$$b_n = \frac{2}{\pi} \int_0^\pi \cos(x) \sin(2nx) dx.$$

Using integration by parts twice, we find that

$$\int_0^\pi \cos(x) \sin(2nx) dx = \frac{4n}{4n^2 - 1}.$$

- **7.30** When we extend  $f(x)$  so that the period is  $2L = \pi$ , the resulting function is odd. Therefore, its Fourier series is just a sine series, and, in fact, is the same series we obtained in the previous exercise.
- **7.42** We need to find constants  $a_0, a_1, a_2, a_3, a_4,$  and  $a_5$  so that

$$\int_{-1}^1 a_0(a_1 + a_2x) dx = 0,$$

$$\int_{-1}^1 a_0(a_3 + a_4x + a_5x^2) dx = 0,$$

and

$$\int_{-1}^1 (a_1 + a_2x)(a_3 + a_4x + a_5x^2) dx = 0.$$

Notice that we can assume, for simplicity, that  $a_0 = a_2 = a_5 = 1$ , and then normalize at the end. We want

$$0 = \int_{-1}^1 (a_1 + x) dx = a_1x|_{-1}^1 + x^2|_{-1}^1 = 2a_1 + 0,$$

so  $a_1 = 0$ . Also, we want

$$0 = \int_{-1}^1 (a_3 + a_4x + x^2) dx = 2a_3 + \frac{2}{3},$$

so  $a_3 = -\frac{1}{3}$ . Finally, we want

$$0 = \int_{-1}^1 x\left(\frac{-1}{3} + a_4x + x^2\right) dx = \frac{2}{3}a_4,$$

so  $a_4 = 0$ . The three orthogonal polynomials are then  $1, x,$  and  $x^2 - \frac{1}{3}$ , or any scalar multiples of these. Now we normalize, to get an orthonormal set.

The first polynomial is a constant,  $P_1(x) = c$ , with

$$\int_{-1}^1 c^2 dx = 1.$$

It follows that  $c = \frac{1}{\sqrt{2}}$ , so that

$$P_1(x) = \frac{1}{\sqrt{2}}.$$

We have

$$\int_{-1}^1 x^2 dx = \frac{2}{3},$$

so the second polynomial is

$$P_2(x) = \sqrt{\frac{3}{2}}x.$$

Finally,

$$\int_{-1}^1 \left(x^2 - \frac{1}{3}\right)^2 dx = \frac{8}{45},$$

so that the third polynomial in the orthonormal family is

$$P_3(x) = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right).$$

- **7.46** Suppose that  $r = (a, b, c)$ . Then  $a = r \cdot i$  and  $b = r \cdot j$ , so

$$(r \cdot i)^2 + (r \cdot j)^2 = a^2 + b^2 \leq a^2 + b^2 + c^2 = |r|^2.$$

- **7.48** We multiply out

$$F(c_1, \dots, c_M) = \int_a^b \left[ f(x) - \sum_{n=1}^M c_n \phi_n(x) \right]^2 dx$$

and use the fact that

$$\int_a^b \phi_n(x) \phi_m(x) dx = 0,$$

if  $m$  and  $n$  are not the same, and equals one if they are, to get

$$F(c_1, \dots, c_M) = \int_a^b f(x)^2 dx - 2 \sum_{n=1}^M c_n \int_a^b f(x) \phi_n(x) dx + \sum_{n=1}^M c_n^2.$$

Since this is a function of the  $M$  variables  $c_1, \dots, c_M$ , we set to zero the partial derivatives of this function with respect to each of the  $c_n$ . Then we have

$$0 = -2 \int_a^b f(x) \phi_n(x) dx + 2c_n,$$

so that

$$c_n = \int_a^b f(x) \phi_n(x) dx.$$

- **7.49** Using integration by parts as many times as needed, show that

$$\int_0^\infty x^n e^{-x} dx = n!,$$

for  $n = 0, 1, \dots$ . Now we show, for example, that

$$0 = \int_0^\infty (1-x)(2-4x+x^2)e^{-x} dx.$$

This becomes

$$0 = \int_0^\infty 2e^{-x} - 6xe^{-x} + 5x^2e^{-x} - x^3e^{-x} dx,$$

or

$$0 = 2(0!) - 6(1!) + 5(2!) - 1(3!) = 2 - 6 + 10 - 6,$$

which is true. The other calculations are similar.