• **16(a):** Note that \( f(x) \) is an even function, so

\[
F(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^1 (1 - x^2) \cos \alpha x \, dx.
\]

Use integration by parts twice to show that

\[
\int_0^1 x^2 \cos \alpha x \, dx = \frac{\sin \alpha}{\alpha} + \frac{2 \cos \alpha}{\alpha^2} - \frac{2 \sin \alpha}{\alpha^3}.
\]

Then

\[
F(\alpha) = 2\sqrt{\frac{2}{\pi}} \sin \alpha - \alpha \cos \alpha + \frac{2 \sin \alpha}{\alpha^2} - \frac{2 \sin \alpha}{\alpha^3}.
\]

Note that the book’s answer is the negative of this answer.

• **16(b):** In 16(a) we learned that the Fourier transform of the function \( f(x) \) that is \( 1 - x^2 \) for \(-1 \leq x \leq 1\), and zero elsewhere is

\[
F(\alpha) = 2\sqrt{\frac{2}{\pi}} \sin \alpha - \alpha \cos \alpha + \frac{2 \sin \alpha}{\alpha^2} - \frac{2 \sin \alpha}{\alpha^3};
\]

note that the book’s answer has the minus sign in the wrong place. It follows from the inversion formula that

\[
2\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{-\alpha \cos \alpha + \sin \alpha}{\alpha^3} \exp(i\alpha) \, d\alpha = \frac{2}{\pi} \int_{-\infty}^\infty \frac{-\alpha \cos \alpha + \sin \alpha}{\alpha^3} \exp(i\alpha) \, d\alpha = f(x).
\]

Therefore,

\[
\frac{4}{\pi} \int_0^\infty \frac{-\alpha \cos \alpha + \sin \alpha}{\alpha^3} \exp(i\frac{\alpha}{2}) \, d\alpha = \frac{2}{\pi} \int_{-\infty}^\infty \frac{-\alpha \cos \alpha + \sin \alpha}{\alpha^3} \exp(i\frac{\alpha}{2}) \, d\alpha = f(1/2) = 3/4.
\]

The book’s answer appears to be missing a minus sign.

• **8.18 (a):** We need to calculate

\[
F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin(\alpha x) \, dx.
\]
Rewriting this as
\[ \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(\alpha x)e^{-x}dx, \]
we should recognize this as the value, at \( s = 1 \), of \( \sqrt{\frac{2}{\pi}} \) times the Laplace transform of the function \( \sin(\alpha x) \). Since the Laplace transform of \( \sin(\alpha x) \) is
\[ F(s) = \frac{\alpha}{s^2 + \alpha^2}, \]
we have
\[ \sqrt{\frac{2}{\pi}} F(1) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + 1}. \]
Now, in preparation for part (b), notice that the inversion formula for the Fourier sine transform, as given in Equation (10), tells us that
\[ f(x) = e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\alpha) \sin(\alpha x)d\alpha \]
\[ = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + 1} \sin(\alpha x)d\alpha = \frac{2}{\pi} \int_0^\infty \frac{\alpha}{\alpha^2 + 1} \sin(\alpha x)d\alpha. \]
Therefore, for part (b), we have
\[ \int_0^\infty \frac{x \sin(mx)}{x^2 + 1} dx = \frac{\pi}{2} f(m) = \frac{\pi}{2} e^{-m}. \]
For part (c), we need to remember that, when we do the Fourier sine transform of a function initially defined for \( x > 0 \), we begin by taking its odd extension; this is the function \( f(x) \) recovered using Equation (10). The odd extension of our original function \( f(x) = e^{-x} \) is discontinuous at \( x = 0 \) and the value given by the second integral in Equation (10) is not \( e^{-0} = 1 \), but the average of \(+1\) and \(-1\), or zero, which is clearly the value of the integral for \( m = 0 \).

- **8.21(a):** The Laplace transform, at \( s = 1 \), of the function \( \cos(\alpha x) \) is
\[ \int_0^\infty e^{-x} \cos(\alpha x)dx = \frac{1}{\alpha^2 + 1}. \]
Therefore, if \( f(x) = e^{-x} \), for \( x > 0 \), and zero otherwise, its Fourier cosine transform is
\[ F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos(\alpha x)dx = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + 1}. \]
By Equation (14), we have
\[ \int_0^\infty F_c(\alpha)^2d\alpha = \int_0^\infty f(x)^2dx. \]
Therefore,
\[ \frac{2}{\pi} \int_0^\infty \frac{1}{\alpha^2 + 1}d\alpha = \int_0^\infty (e^{-x})^2dx = \frac{1}{2}. \]
8.23: We saw, in Problem 8.16(a), that, if \( f(x) \) is the function that is \( 1 - x^2 \), for \(|x| \leq 1\), and zero, otherwise, then its Fourier transform is the function

\[
F(\alpha) = 2 \sqrt{\frac{2}{\pi}} \frac{-\alpha \cos(\alpha) + \sin(\alpha)}{\alpha^3}.
\]

Now apply Equation (15), with \( G(\alpha) = F(\alpha) \). We then have

\[
\int_0^{\infty} \left( \frac{\alpha \cos(\alpha) - \sin(\alpha)}{\alpha^3} \right)^2 d\alpha = \frac{1}{2} \int_{-\infty}^{\infty} \left( \frac{\alpha \cos(\alpha) - \sin(\alpha)}{\alpha^3} \right)^2 d\alpha
\]

\[
= \frac{1}{2} \frac{\pi}{8} \int_{-\infty}^{\infty} F(\alpha)^2 d\alpha = \frac{\pi}{16} \int_{-1}^{1} (1 - x^2)^2 dx = \frac{\pi}{15}.
\]