92.530 Applied Mathematics I: Solutions to Homework Problems in Chapter 8

- **16(a):** Note that $f(x)$ is an even function, so

$$F(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^1 (1 - x^2) \cos \alpha x \, dx.$$

Use integration by parts twice to show that

$$\int_0^1 x^2 \cos \alpha x \, dx = \frac{\sin \alpha}{\alpha} + \frac{2 \cos \alpha}{\alpha^2} - \frac{2 \sin \alpha}{\alpha^3}.$$

Then

$$F(\alpha) = 2\sqrt{\frac{2}{\pi}} \frac{\sin \alpha - \alpha \cos \alpha}{\alpha^3}.$$

Note that the book's answer is the negative of this answer.

- **16(b):** In 16(a) we learned that the Fourier transform of the function $f(x)$ that is $1 - x^2$, for $-1 \le x \le 1$, and zero elsewhere is

$$F(\alpha) = 2\sqrt{\frac{2}{\pi}} \frac{-\alpha \cos \alpha + \sin \alpha}{\alpha^3};$$

note that the book's answer has the minus sign in the wrong place. It follows from the inversion formula that

$$2\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{-\alpha \cos \alpha + \sin \alpha}{\alpha^3} \exp(ix\alpha) \, d\alpha = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{-\alpha \cos \alpha + \sin \alpha}{\alpha^3} \exp(ix\alpha) d\alpha = f(x).$$

Therefore,

$$\frac{4}{\pi} \int_0^{\infty} \frac{-\alpha \cos \alpha + \sin \alpha}{\alpha^3} \exp(i\frac{\alpha}{2}) d\alpha = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{-\alpha \cos \alpha + \sin \alpha}{\alpha^3} \exp(i\frac{\alpha}{2}) \, d\alpha = f(1/2) = 3/4.$$

The book's answer appears to be missing a minus sign.

- **8.18** (a): We need to calculate

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin(\alpha x) dx.$$

Rewriting this as

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \sin(\alpha x)e^{-x}dx,$$

we should recognize this as the value, at $s = 1$, of $\sqrt{\frac{2}{\pi}}$ times the Laplace transform of the function $\sin(\alpha x)$. Since the Laplace transform of $\sin(\alpha x)$ is

$$F(s) = \frac{\alpha}{s^2+\alpha^2},$$

we have

$$\sqrt{\frac{2}{\pi}}F(1) = \sqrt{\frac{2}{\pi}}\frac{\alpha}{\alpha^2+1}.$$

Now, in preparation for part (b), notice that the inversion formula for the Fourier sine transform, as given in Equation (10), tells us that

$$f(x) = e^{-x} = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(\alpha)\sin(\alpha x)d\alpha$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{\alpha}{\alpha^2+1}\sin(\alpha x)d\alpha = \frac{2}{\pi}\int_0^\infty \frac{\alpha}{\alpha^2+1}\sin(\alpha x)d\alpha.$$

Therefore, for part (b), we have

$$\int_0^\infty \frac{x\sin(mx)}{x^2+1}dx = \frac{\pi}{2}f(m) = \frac{\pi}{2}e^{-m}.$$

For part (c), we need to remember that, when we do the Fourier sine transform of a function initially defined for $x > 0$, we begin by taking its odd extension; this is the function $f(x)$ recovered using Equation (10). The odd extension of our original function $f(x) = e^{-x}$ is discontinuous at $x = 0$ and the value given by the second integral in Equation (10) is not $e^{-0} = 1$, but the average of $+1$ and -1, or zero, which is clearly the value of the integral for $m = 0$.

- **8.21(a):** The Laplace transform, at $s = 1$, of the function $\cos(\alpha x)$ is

$$\int_0^\infty e^{-x}\cos(\alpha x)dx = \frac{1}{\alpha^2+1}.$$

Therefore, if $f(x) = e^{-x}$, for $x > 0$, and zero otherwise, its Fourier cosine transform is

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos(\alpha x)dx = \sqrt{\frac{2}{\pi}}\frac{1}{\alpha^2+1}.$$

By Equation (14), we have

$$\int_0^\infty F_c(\alpha)^2 d\alpha = \int_0^\infty f(x)^2 dx.$$

Therefore,

$$\frac{2}{\pi}\int_0^\infty (\frac{1}{\alpha^2+1})^2 d\alpha = \int_0^\infty (e^{-x})^2 dx = \frac{1}{2}.$$

- **8.23:** We saw, in Problem 8.16(a), that, if $f(x)$ is the function that is $1-x^2$, for $|x| \leq 1$, and zero, otherwise, then its Fourier transform is the function

$$F(\alpha) = 2\sqrt{\frac{2}{\pi}}\frac{-\alpha\cos(\alpha)+\sin(\alpha)}{\alpha^3}.$$

Now apply Equation (15), with $G(\alpha) = F(\alpha)$. We then have

$$\int_0^\infty \Big(\frac{\alpha\cos(\alpha)-\sin(\alpha)}{\alpha^3}\Big)^2 d\alpha = \frac{1}{2}\int_{-\infty}^\infty \Big(\frac{\alpha\cos(\alpha)-\sin(\alpha)}{\alpha^3}\Big)^2 d\alpha$$

$$= \frac{1}{2}\frac{\pi}{8}\int_{-\infty}^\infty F(\alpha)^2 d\alpha = \frac{\pi}{16}\int_{-1}^1 (1-x^2)^2 dx = \frac{\pi}{15}.$$