92.531 Applied Mathematics II: Solutions to Homework Problems in Chapter 9

- **9.29** (a): We have

$$\int_0^\infty x^4 e^{-x} dx = \int_0^\infty e^{-x} x^{5-1} dx = \Gamma(5) = 4! = 24.$$

(b): Using $t = 3x$, we have

$$\int_0^\infty e^{-3x} x^6 dx = \frac{1}{3} \int_0^\infty e^{-t} \Big(\frac{t}{3}\Big)^6 dt$$

$$= 3^{-7} \int_0^\infty e^{-t} t^{7-1} dt = 3^{-7} \Gamma(7) = \frac{6!}{3^7} = \frac{80}{243}.$$

(c): Using $t = 2x^2$, so that $dt = 4x dx$, we have

$$\int_0^\infty x^2 e^{-2x^2} dx = \frac{1}{4\sqrt{2}} \int_0^\infty e^{-t} t^{\frac{3}{2}-1} dt = \frac{1}{4\sqrt{2}} \Gamma(\frac{3}{2}) = \frac{\sqrt{2\pi}}{16}.$$

- **9.40** (a):

$$\int_0^1 x^2 (1-x)^3 dx = B(3,4) = \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{2!3!}{6!} = \frac{1}{60}.$$

(b):

$$\int_0^1 \sqrt{\frac{1-x}{x}} dx = \int_0^1 (1-x)^{1/2} x^{-1/2} dx = B(\frac{1}{2}, \frac{3}{2}) = \frac{\pi}{2}.$$

(c): Using $t = x^2/4$, so that $dx = t^{-1/2} dt$, we have

$$\int_0^2 (4-x^2)^{3/2} dx = 8 \int_0^1 (1-t)^{3/2} t^{-1/2} dt = 8B(\frac{5}{2}, \frac{1}{2}) = 3\pi.$$

- **9.49** This problem is two-dimensional and does not fit into the Dirichlet integral formula given in the text. Instead, we just treat it as an ordinary double integral,

$$\int_0^1 \int_0^{1-x} x^{1/2} y^{1/2} dy\, dx = \frac{2}{3} \int_0^1 x^{1/2} (1-x)^{1/2} dx = \frac{2}{3} B(3/2, 5/2) = \pi/24.$$

There is a more general version of the Dirichlet integral formulas that we could also use here. This more general Dirichlet integral formula tells us that, for

$$I = \int\int\cdots\int f(t_1 + t_2 + ... + t_n) t_1^{a_1-1}\cdots t_n^{a_n-1} dt_1 dt_2 ... dt_n,$$

with the integral over the region bounded by the positive axes and the surface $\sum_{i=1}^{n} t_i = 1$, we have

$$I = \frac{\Gamma(a_1)\cdots\Gamma(a_n)}{\Gamma(a_1 + ... + a_n)} \int_0^1 f(u) u^{a_1+...+a_n-1} du.$$

Therefore, in our problem here, since we have $n = 2$, $f(x) = 1$, and $a_1 = a_2 = 3/2$, we have

$$I = \frac{\Gamma(3/2)\Gamma(3/2)}{\Gamma(3)} \int_0^1 u^2 du = \frac{\pi}{24}.$$