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COMPLETION OF LATTICES OF SEMI-CONTINUOUS FUNCTIONS

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Abstract

If U and V are topologies on an abstract set X, then the triple (X, U, V) is a bitopological space. Using the theorem of Priestley on the representation of distributive lattices, results of Dilworth concerning the normal completion of the lattice of bounded, continuous, real-valued functions on a topological space are extended to include the lattice of bounded, semi-continuous, real-valued functions on certain bitopological spaces. The distributivity of certain lattices is investigated, and the theorem of Funayama on distributive normal completions is generalized.

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1. Introduction and notation

If (X, U) is a topological space that is not completely regular, the collection of continuous real-valued functions defined on X may contain only constant functions. The results of Wilson (1931) and Csaszar (1960) reveal that, in this case, it is wise to introduce a second topology, say V, and consider the richer collection of functions from X to the reals that are U-upper semi-continuous and V-lower semi-continuous. Wilson has shown that the lack of symmetry that results when U is generated by a quasi-metric (a metric lacking the symmetric property) can be somewhat overcome by the introduction of a closely related second topology. Csaszar, in a similar manner, shows that poor separation structure in U can be replaced by good

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"bitopological" separation structure, when we consider a suitably chosen bitopological space, $(X, \mathbf{U}, \mathbf{V})$. In this article, we shall concern ourselves with the collection of all bounded U-upper semi-continuous, V-lower semi-continuous functions from $(X, \mathbf{U}, \mathbf{V})$ to the reals, studying, in particular, the lattice structure of this collection, which we denote by $S(X)$.

Dilworth (1950) investigates the lattice of all bounded continuous real-valued functions on a topological space, obtaining a representation for the normal completion of this lattice, using Stone's representation theorem for Boolean algebras and the notion of extremally disconnected space. Recently, Datta (1972) has introduced the analogous notion of pairwise extremally disconnected bitopological space, and Priestley (1970) has obtained an elegant representation theorem for distributive lattices, resembling the construction of Stone, and involving bitopological spaces. In this paper, we extend Dilworth's results to the study of the lattice $S(X)$, using both the extremally disconnected spaces of Datta and the representation theorem of Priestley. As will be clear later, any such extension of Dilworth's results must involve a discussion of distributivity of certain lattices. Indeed, as a consequence of our investigations, we obtain a generalization of the result of Funayama (1944) on distributive normal completions of lattices and a somewhat different view of the theorem itself.

Throughout this paper X will denote a set, $\mathbf{U}$ and $\mathbf{V}$ topologies on X, with $\mathbf{T}$ the smallest topology containing both $\mathbf{U}$ and $\mathbf{V}$. By $B(X)$ we denote the collection of all bounded real-valued functions on X, and by $C(X)$, the continuous members of $B(X)$, where the topology on X is understood from the context.

2. Semi-continuous functions and normal functions

A function f in $B(X)$ is U-upper semi-continuous (U-usc) if, for all real t, $\{x \mid f(x) < t\}$ is in $\mathbf{U}$, and V-lower semi-continuous (V-lsc) if, for all real t, $\{x \mid f(x) > t\}$ is in $\mathbf{V}$.

A bitopological space $(X, \mathbf{U}, \mathbf{V})$ is pairwise completely regular (see Reilly (1972)) if the following condition, and the one obtained by interchanging the roles of $\mathbf{U}$ and $\mathbf{V}$, obtain: for every U-closed set K and x not in K there is a U-usc, V-lsc f in $B(X)$ with $f(x) = 0$ and f identically 1 on K. Throughout this paper we shall assume that $(X, \mathbf{U}, \mathbf{V})$ is pairwise completely regular.

A bitopological space $(X, \mathbf{U}, \mathbf{V})$ is pairwise regular if for every U-closed (V-closed) set K and x not in K, there are disjoint sets U in $\mathbf{U}$ and V in $\mathbf{V}$ with x in U and $K \subseteq V$ (x in V and $K \subseteq U$). Every pairwise completely regular space is pairwise regular.

We now define the notion of normal function. For any h in $B(X)$, let

$$h^*(x) = \inf_{U(x)} \sup_{y \in U(x)} h(y), \quad h_*(x) = \sup_{V(x)} \inf_{y \in V(x)} h(y),$$

where $U(x)$ and $V(x)$ represent arbitrary **U** and **V** neighborhoods of x. Then we have that $h^* = (h^*)^*$ and $h_* = (h_*)_*$ for any h in $B(X)$, and that $h = h^*$ if and only if h is **U**-usc and $h = h_*$ if and only if h is **V**-lsc. We say that h in $B(X)$ is normal if $h = (h^*)_*$. This implies, obviously, that a normal h is also **V**-lsc. The collection of all normal functions in $B(X)$ we denote by $N(X)$. This collection is a lattice under the following operations: for f and h in $N(X)$ let

$$f \vee h = ([\sup\{f, h\}]^*)_*, \quad f \wedge h = \inf\{f, h\},$$

where sup and inf are the pointwise supremum and infimum. For infinite collections we have

$$\bigvee_\alpha h_\alpha = ([\sup\{h_\alpha\}]^*)_*, \quad \bigwedge_\alpha h_\alpha = [\inf\{h_\alpha\}]_*.$$

A subset E of X is called pairwise regular open (relative to **U** and **V**, in that order) if $E = \mathbf{V}\text{-int}(\mathbf{U}\text{-cl}(E))$. If **S** is the collection of all such subsets of X, then **S** is a lattice under the following operations: for E and F in **S**, let

$$E \vee F = \mathbf{V}\text{-int}(\mathbf{U}\text{-cl}(E \cup F)), \quad E \wedge F = E \cap F.$$

For infinite collections $\{E_\alpha\}$, we let

$$\bigvee_\alpha \{E_\alpha\} = \mathbf{V}\text{-int}(\mathbf{U}\text{-cl}(\bigcup_\alpha E_\alpha)), \bigwedge_\alpha \{E_\alpha\} = \mathbf{V}\text{-int}(\bigcap_\alpha E_\alpha).$$

Via the association of E with the characteristic function of the set E, χ_E, we embed the lattice **S** isomorphically into the lattice $N(X)$. In other words, E is in **S** if and only if the function χ_E that is 1 on E and 0 off E is in $N(X)$.

3. The normal completion of $S(X)$

If $(L, <)$ is any lattice and $M \subseteq L$ a subset, we let M^* be the set of upper bounds of M, and M_* the set of lower bounds of M. Then M is a normal subset ("closed" is Birkhoff's terminology) of L if and only if $M = (M^*)_*$. If l_0 is a member of L, then the subset $[l_0] = \{l \mid l \leqslant l_0\}$ is normal and is a principal normal subset. If we let L' be the collection of all normal subsets of L, then the operations $K \vee M = ((K \cup M)^*)_*$, $K \wedge M = K \cap M$ provide L' with lattice structure, and via the association of l_0 with $[l_0]$, L is isomorphically a sub-lattice of L'. It is well known that L' is then the minimal completion of L. Funayama has shown (1944) that it is possible for L to be distributive while L' is not even modular. We shall discuss this in detail later.

Following Dilworth's lead, we shall now show that the normal completion of the lattice $S(X)$ can be concretely realized as the lattice $N(X)$, and also obtain a necessary and sufficient condition for $S(X)$ to be $N(X)$. We have modified

Dilworth's notation somewhat but the proofs are quite similar and are omitted. In the next section we shall focus on the last result in Dilworth's article, a result that does not extend so easily to our case. Our efforts lead us to a discussion of distributivity, and once certain assumptions are made, we do succeed in extending this result. The proofs there differ from those of Dilworth and will be presented in full.

PROPOSITION 3.1. *Let f be* **V**-*lsc in* $B(X)$. *Then f is normal if and only if, for all real t, the set* $\{x \mid f(x) < t\}$ *is the union of sets whose complements are in* **S**.

COROLLARY 1. *Every member of* $N(X)$ *is in* $S(X)$ *if and only if the* **V**-*closure of each* **U**-*open set is again* **U**-*open.*

REMARK. Datta (1972) calls a bitopological space $(X, \mathbf{U}, \mathbf{V})$ pairwise extremally disconnected if the **V**-closure of a **U**-open set is always **U**-open (or, equivalently, if the **U**-closure of a **V**-open set is always **V**-open). This notion is also discussed in Priestley (1972).

If h is in $B(X)$, let $(h) = \{f \text{ in } S(X) \mid f(x) \leqslant h(x) \text{ for all } x\}$. One can show then that $h_* = \sup\{f \text{ in } (h)\}$. Furthermore, if h is normal then $h = \sup\{f \text{ in } (h)\}$ and the set (h) is normal in the lattice $S(X)$. The proofs of these assertions follow those of Dilworth (1950) and are not given here. The association of h in $N(X)$ with (h), a normal subset of $S(X)$, establishes a lattice isomorphism between $N(X)$ and the normal completion of $S(X)$. Hence we can extend the above corollary to

COROLLARY 2. *The lattice* $S(X)$ *is complete if and only if* $(X, \mathbf{U}, \mathbf{V})$ *is pairwise extremally disconnected.*

In his article, Dilworth goes one step further and shows that every $N(X)$ is isomorphic to a $C(Y)$ for some extremally disconnected topological space Y. This cannot be the case, generally, for bitopological spaces. As we have seen, **S** is a sub-lattice of $N(X)$. If $N(X)$ were isomorphic to $S(Y)$ in every case, then $N(X)$, and therefore **S**, would be distributive, since the lattice operations in $S(Y)$ are pointwise ones. However, we present below an example in which **S** is not distributive. As we shall presently see, though, the distributivity of **S**, in addition to being necessary for the extension of the Dilworth result, is, in fact, sufficient.

4. $N(X) = S(Y)$ if and only if S is distributive

Suppose L is a distributive lattice (with a largest and a smallest member, denoted 0 and 1) and let Y be the collection of all $\{0, 1\}$-valued lattice homomorphisms on L.

Viewing Y as a subset of the product of L copies of $\{0, 1\}$, we give Y the product topology stemming from the discrete topology on $\{0, 1\}$. We order Y by saying $y_1 \leqslant y_2$ if and only if $y_1(l) \leqslant y_2(l)$ for all l in L. Let $\mathbf{D}$ be all open decreasing subsets of Y, and $\mathbf{I}$ all the open increasing subsets of Y, where I is increasing if and only if $y \in I$ and $z \geqslant y$ implies $z \in I$. The space $(Y, \mathbf{D}, \mathbf{I})$ is then a bitopological space and it is Priestley's theorem (1970) that the lattice L is isomorphic to the lattice of $\mathbf{D}$-closed, $\mathbf{I}$-open subsets of Y, via the association of l in L with $\{y \mid y(l) = 1\}$.

If M is any subset of L, then $\{y \mid y(l) = 1$ for some $l \in M\}$ is clearly $\mathbf{I}$-open. If J is the ideal in L generated by M, then

$$\{y \mid y(l) = 1 \text{ for some } l \in M\} = \{y \mid y(l) = 1 \text{ for some } l \in J\}$$

since if y is 0 for every member of M, then y is 0 for every element of L that is less than a member of M, and also for any finite suprema formed from these elements. Hence y is 0 on all members of J. Therefore, an arbitrary $\mathbf{I}$-open set I has the form $I = \{y \mid y(l) = 1$ for some l in $J\}$ where J is an ideal in L. Similarly, D is in $\mathbf{D}$ if and only if $D = \{y \mid y(l) = 0$ for some l in $K\}$ where K is a filter in L. If I is in $\mathbf{I}$, then $\mathbf{D}\text{-cl}(I) = \{y \mid y(l) = 1$ for all l in $J^*\}$ and $\mathbf{I}\text{-int}(\mathbf{D}\text{-cl}(I))$ is equal to

$$\{y \mid y(l) = 1 \text{ for some } l \text{ in } (J^*)_*\}.$$

Therefore, a subset I of Y has the property that $I = \mathbf{I}\text{-int}(\mathbf{D}\text{-cl}(I))$ if and only if I can be represented as $I = \{y \mid y(l) = 1\}$, for some l in M, where $(M^*)_* = M$, that is, M is a normal subset of L.

In the case considered by Dilworth, the regular open subsets formed a Boolean algebra, and the natural symmetry of this structure facilitated a number of the proofs. Here we assume only that $\mathbf{S}$ is a distributive lattice, and let $(Y, \mathbf{D}, \mathbf{I})$ be the representation space for $\mathbf{S}$. Since $\mathbf{S}$ is a complete lattice, the space $(Y, \mathbf{D}, \mathbf{I})$ will be pairwise extremally disconnected. We shall show that $N(X)$ and $S(Y)$ are lattice isomorphic. To compensate for the lack of symmetry, we define two mappings from $N(X)$ to $S(Y)$ and two from $S(Y)$ to $N(X)$. After showing that in each case the two are really the same, we will have the option of using either of the two in later proofs. Notice that $S(Y)$ consists of the continuous increasing functions on Y.

Definition 4.1. Let σ: $B(X) \to B(Y)$ be given by

$$(\sigma f)(y) = \sup\{\inf_{x \in E} f(x) \colon E \in \mathbf{S}, y(E) = 1\}.$$

Definition 4.2. Let $\bar{\sigma}$: $B(X) \to B(Y)$ be given by

$$(\bar{\sigma} f)(y) = \inf\{\sup_{x \notin E} f(x) \colon E \in \mathbf{S}, y(E) = 0\}.$$

DEFINITION 4.3. Let τ: $B(Y)\to B(X)$ be given by

$$(\tau F)(x) = \inf\{\sup_{y:\, y(E)=0} F(y) \colon E\in \mathbf{S}, x\notin E\}.$$

DEFINITION 4.4. Let $\bar{\tau}$: $B(Y)\to B(X)$ be given by

$$(\bar{\tau}F)(x) = \sup\{\inf_{y:\, y(E)=1} F(y) \colon E\in \mathbf{S}, x\in E\}$$

PROPOSITION 4.1. *If f is in $B(X)$ then $\bar{\sigma}f$ is* **D**-*usc, and σf is* **I**-*lsc. If f is in $N(X)$, then $\sigma f = \bar{\sigma}f$.*

PROOF. Let $Z = \{y \mid \bar{\sigma}f(y) < a\}$, and let $z\in Z$. Then there is an E in **S** with $z(E) = 0$ and $\sup f(x) < a$, the supremum taken over all x not in E. Let $D = \{y \mid y(E) = 0\}$. Then $D\in \mathbf{D}$ and $\bar{\sigma}f(y) < a$ for all y in D. So Z is **D**-open. Similarly we can show σf to be **I**-lsc.

Suppose now that f is in $N(X)$. If, for some z in Y we have $\bar{\sigma}f(z) < t < \sigma f(z)$, then there are E and B in **S**, with $z(E) = 0$, $z(B) = 1$, and the supremum of f on $X-E$ less than the infimum of f on B. Consequently $B\subseteq E$, a contradiction, since $z(E) = 0$ while $z(B) = 1$. So $\sigma f \leqslant \bar{\sigma}f$.

If, for some $w\in Y$ we have $\bar{\sigma}f(w) > t > \sigma f(w)$, then let $C = \{x \mid f(x)\geqslant t\}$. By Proposition 2.1, we know that there are sets E_α in **S** with $C = \bigcap_\alpha E_\alpha$. Therefore $C = \bigcap_\alpha \mathbf{V}\text{-int}(\mathbf{U}\text{-cl}(E_\alpha)) \subseteq \bigcap_\alpha \mathbf{U}\text{-cl}(E_\alpha)$. Also

$$\mathbf{V}\text{-int}(\textstyle\bigcap_\alpha \mathbf{U}\text{-cl}(E_\alpha)) \subseteq \bigcap_\alpha \mathbf{V}\text{-int}(\mathbf{U}\text{-cl}(E_\alpha)) = C.$$

Let $A = \mathbf{V}\text{-int}\,\mathbf{U}\text{-cl}\,C$. Then

$$\emptyset \neq \{x \mid f(x) > t\} \subseteq \mathbf{V}\text{-int}(C) \subseteq A \subseteq \mathbf{V}\text{-int}(\textstyle\bigcap_\alpha \mathbf{U}\text{-cl}(E_\alpha) \subseteq C \neq X.$$

So A is a member of **S** and is contained in C. It must then be the case that $w(A) = 1$, for if $w(A) = 0$ then $\bar{\sigma}f(w) \leqslant \sup f(x)$, taken over x not in A, this supremum being, at most, t. Since $\sigma f(x) < t$ and $w(A) = 1$, it follows that the infimum of $f(x)$, over x in A, is less than t, a contradiction, since $A\subseteq C$. Therefore, σ and $\bar{\sigma}$ agree on $N(X)$.

PROPOSITION 4.2. *If F is in $B(Y)$, then $\bar{\tau}F$ is* **V**-*lsc. Furthermore, if F is in $S(Y)$, then τF is normal and equal to $\bar{\tau}F$.*

PROOF. The first assertion is easily proved and we omit it here. We start by assuming that F is in $S(Y)$. If, for some x in X, we have $\tau F(x) < \bar{\tau}F(x)$, then there are E and B in **S**, with x in E but not in B, and with $\sup F(y)$, over y with $y(B) = 0$, strictly less than $\inf F(y)$, over y with $y(E) = 1$. Consequently, there is no y with

$y(B)=0$ and $y(E)=1$. It follows, then, that $E \leqslant B$ in $\mathbf{S}$, or $E \subseteq B$. This is a contradiction, from which we conclude that $\bar{\tau}F \leqslant \tau F$. If, for some $\bar{x}$, we have

$$\bar{\tau}F(\bar{x}) = p < t < r = \tau F(\bar{x}),$$

then for every A in $\mathbf{S}$ with $\bar{x} \in A$, we have $\inf F(y) \leqslant p < t$, with the infimum over all y with $y(A)=1$. Let $D = \{y \mid F(y) < t\}$. Since F is in $S(Y)$, it is $\mathbf{D}$-usc and hence D is in $\mathbf{D}$. Since $(Y, \mathbf{D}, \mathbf{I})$ is pairwise extremally disconnected, we have $G = \mathbf{I}\text{-cl}(D)$ again in $\mathbf{D}$. Therefore G is $\mathbf{I}$-closed and $\mathbf{D}$-open. According to the Priestley representation theorem, there is an E in $\mathbf{S}$ such that $G = \{y \mid y(E)=0\}$. If $\bar{x}$ were in E, we would have $F(\bar{y}) \leqslant p < t$ for some $\bar{y}$ such that $\bar{y}(E)=1$. Since $F(\bar{y}) < t$, $\bar{y}$ is in D, hence in G, and so $\bar{y}(E)=0$, a contradiction. So $\bar{x}$ is not in E. Therefore, $r < \sup F(y)$, over all y with $y(E)=0$, and there is a y' with $F(y') > r$ and $y'(E)=0$. It follows that G intersects $\{y \mid F(y) > r\}$. But the set $\{y \mid F(y) \leqslant t\}$ is $\mathbf{I}$-closed and contains D, hence contains G. So τ and $\bar{\tau}$ agree on members of $S(Y)$. We need only prove that τF is normal. Consider the set $Z = \{x \mid \tau F(x) < a\}$, and z in Z. Let $\tau F(z) = b < a$. For some E in $\mathbf{S}$ that does not contain z, we have $\sup F(y) < a$, with the supremum over all y with $y(E)=0$. We show that $X - E$ is a subset of Z. If not, then there is $\bar{x}$ not in E and not in Z. So $\tau F(\bar{x}) \geqslant a$. Since $\bar{x}$ is not in E, $\sup F(y)$, over y with $y(E)=0$, is greater than $\tau F(\bar{x})$, which is greater than or equal to a, a contradiction. Hence Z is the union of sets whose complements are members of $\mathbf{S}$, so τF is normal by Proposition 3.1.

With the help of several lemmas, we shall establish that $\tau: S(Y) \to N(X)$ and $\sigma: N(X) \to S(Y)$ are mutually inverse lattice isomorphisms.

LEMMA 4.1. *For every E in $\mathbf{S}$ that contains x, and for every f in $B(X)$, $\tau\sigma f(x) \geqslant \inf f(z)$, taken over all $z \in E$.*

PROOF. For every E containing x, we have $\tau\sigma f(x) \geqslant \inf \sigma f(y)$, the infimum taken over all y with $y(E)=1$. Hence, for each such E, there is y, with $y(E)=1$ and $\tau\sigma f(x) \geqslant \sigma f(y)$. Clearly $\sigma f(y) \geqslant \inf f(z)$, over all z in E.

LEMMA 4.2. *If f is in $B(X)$, then $\tau\sigma f \geqslant f_*$.*

PROOF. If we have $\tau\sigma f(x) < t < f_*(x)$ for some x, then, by the definition of f_*, there is a $\mathbf{V}$-neighborhood of x with $\inf f(z)$, taken over z in this neighborhood, greater than t. Since $(X, \mathbf{U}, \mathbf{V})$ is pairwise regular, we can assume that this $\mathbf{V}$-neighborhood is actually a member of $\mathbf{S}$, and we call it E. Since E contains x, we know, from the previous lemma, that $\tau\sigma f(x) \geqslant \inf f(z)$, the infimum taken over z in E. So $\tau\sigma f(x) \geqslant t$, a contradiction.

LEMMA 4.3. *If f is* $\mathbf{V}$*-lsc in* $B(X)$, *then* $\tau\bar{\sigma}f \leqslant f^*$.

PROOF. If we have $\tau\bar{\sigma}f(x) > t > f^*(x)$, then there is a $\mathbf{U}$-neighborhood, $U(x)$, of x, with $\sup f(z) < t$, the supremum taken over all z in $U(x)$. By the pairwise regularity of $(X, \mathbf{U}, \mathbf{V})$ we may assume that $U(x)$ is the $\mathbf{U}$-interior of its $\mathbf{V}$-closure. Then we let $E = X - \mathbf{V}\text{-cl}(U(x))$. Then E is in $\mathbf{S}$ and does not contain x. Since $\tau\bar{\sigma}f(x) > t$, we have $\sup \bar{\sigma}f(y) > t$, where this supremum is taken over all y with $y(E) = 0$. We then can select a $y \in Y$ with $\bar{\sigma}f(y) > t$ and $y(E) = 0$. Since $\bar{\sigma}f(y) \leqslant \sup f(z)$, over all z not in E, we know that $\bar{\sigma}f(y)$ is not greater than the supremum of f on $\mathbf{V}\text{-cl}(U(x))$. Since f is $\mathbf{V}$-lsc and $U(x) \subseteq f^{-1}(-\infty, t) \subseteq f^{-1}(-\infty, t]$, with the latter $\mathbf{V}$-closed, we know that $\bar{\sigma}f(y) \leqslant t$, a contradiction.

COROLLARY 4.1. *If f is in $N(X)$ then* $f_* \leqslant \tau\sigma f \leqslant f^*$.

LEMMA 4.4. *If F is in $B(Y)$, then* $\bar{\sigma}\tau F \leqslant F^*$.

PROOF. It is clear that $\bar{\sigma}\tau F(y) \leqslant \sup F(w)$, with the supremum over w with $w(E) = 0$, for every E in $\mathbf{S}$ with $y(E) = 0$. If $\bar{\sigma}\tau F(\bar{y}) > t > F^*(\bar{y})$ for some $\bar{y}$, then there is a $\mathbf{D}$-neighborhood, D, of $\bar{y}$, with $\sup F(w) < t$, the supremum over all w in D. Since the $\mathbf{I}$-closed, $\mathbf{D}$-open sets form a base for $\mathbf{D}$, we assume that D is also $\mathbf{I}$-closed and hence has the form $D = \{w \mid w(E) = 0\}$ for some E in $\mathbf{S}$. Then $\bar{y}(E) = 0$, so that $\bar{\sigma}\tau F(\bar{y}) \leqslant \sup F(w)$ over all w in D. Therefore $\bar{\sigma}\tau F(\bar{y}) \leqslant t$, a contradiction.

LEMMA 4.5. *If F is in $B(Y)$ then* $\sigma\bar{\tau}F \geqslant F_*$.

PROOF. The proof is similar to that of Lemma 4.4 and we omit it here.

COROLLARY 4.2. *If F is in $S(Y)$ then* $F_* \leqslant \sigma\tau F \leqslant F^*$.

LEMMA 4.6. *If f is in $N(X)$ then* $\tau\sigma f \leqslant f$.

PROOF. We have $\tau\sigma f = (\tau\sigma f)^*{}_*$ and $\tau\sigma f \leqslant f^*$, so that $(\tau\sigma f)^* \leqslant f^{**} = f^*$ and $(\tau\sigma f)^*{}_* \leqslant f^*{}_* = f$.

LEMMA 4.7. *If F is in $S(Y)$ then* $\sigma\tau F \leqslant F$.

PROOF. It is essentially the same as that of Lemma 4.6 and we omit it.

LEMMA 4.8. *If f and g are in $N(X)$ and $f \geqslant g$ then $\sigma f \geqslant \sigma g$. Similarly, if $F \geqslant G$ in $S(Y)$ then $\tau F \geqslant \tau G$.*

PROOF. Obvious.

We now summarize these results in the following:

THEOREM 4.1. *If f is in $N(X)$ then $\tau\sigma f = f$. If F is in $S(Y)$ then $\sigma\tau F = F$. The lattices $N(X)$ and $S(Y)$ are isomorphic.*

COROLLARY 4.3. *If $(X, \mathbf{U}, \mathbf{V})$ is a pairwise completely regular bitopological space, then the lattice $S(X)$ has a distributive normal completion if and only if the lattice* **S** *is distributive.*

5. Funayama's theorem and the distributivity of S

Let us begin by considering some examples.

EXAMPLE 5.1. Let X be the real line, **U** the topology generated by right infinite open rays, and **V** the topology generated by left infinite open rays. As can be easily verified, the lattice operation $\vee$ on **S** is simply the set-theoretic union, so that **S** is distributive. The topology generated by **U** and **V** is the usual topology.

EXAMPLE 5.2. Let X be the real line, **V** the topology generated by the sets $(a, b]$ and **U** the topology generated by the sets $[c, d)$. Let $E = (-\infty, 1)$, $F = (-\infty, 1]$, and $G = (1, +\infty)$. Each of these sets is in **S**, and $E \vee G = F \vee G = X$, $E \wedge G = F \wedge G = \emptyset$, but $E \neq F$. Therefore **S** is not distributive. Note that the topology generated by **U** and **V** is the discrete topology.

EXAMPLE 5.3. Let $X = \{0, 1, 2, 3, \ldots\}$, **V** the topology generated by the sets whose complements both contain 0 and are finite, and **U** the topology generated by all supersets of 0. Here every member of **V** is in **S** and so the lattice operation $\vee$ is set-theoretic union and **S** is distributive. Note that as in Example 5.2, the topology generated by **U** and **V** is discrete. So this phenomenon is not relevant to the distributivity question.

EXAMPLE 5.4. Let X be the real line, **V** the usual topology on X, and **U** the topology generated by the sets $[c, d)$. Then $\mathbf{U} \supseteq \mathbf{V}$, **S** is not all of **V**, and the lattice operation $\vee$ is not set-theoretic union. However, **S** is distributive as can be verified directly, or derived as a consequence of our theorem below.

From now on, E, F, G and H will denote arbitrary members of **S**. We denote by $A^\wedge$ the **U**-closure of set A, and by $A^\sim$ the **V**-interior.

PROPOSITION 5.1. *The following are equivalent:*
(1) $\mathbf{S}$ *is distributive;*
(2) $E\wedge(F\vee G)=(E\wedge F)\vee(E\wedge G)$ *for all* E,F,G *in* $\mathbf{S}$*;*
(3) $E^{\wedge\sim}\cap(F\cup G)^{\wedge\sim}=(E\cap(F\cup G))^{\wedge\sim}$ *for every* E,F,G *in* $\mathbf{S}$*;*
(4) *for any* H *in* $\mathbf{S}$, $H\subseteq(F\cup G)^{\wedge}$ *implies that* $H^{\wedge}=(H\cap(F\cup G))^{\wedge}$.

PROOF. (1) if and only if (2) is immediate, as is (2) if and only if (3). Assume then that (3) holds, and suppose that $H\subseteq\mathbf{U}$-closure $(F\cup G)$. Since H is in $\mathbf{V}$, we know $H\subseteq H^{\wedge\sim}\cap(F\cup G)^{\wedge\sim}$, so that $H=H^{\wedge\sim}\cap(F\cup G)^{\wedge\sim}$. Since

$$H=H^{\wedge\sim}\cap(F\cup G)^{\wedge\sim}=(H\cap(F\cup G))^{\wedge\sim}$$

by (3), we have that $H^{\wedge}=(H\cap(F\cup G))^{\wedge}$, so that (4) holds. Assume now that (4) holds. Let $H=E\cap(F\cup G)^{\wedge\sim}=E^{\wedge\sim}\cap(F\cup G)^{\wedge\sim}$. Then $H\subseteq(F\cup G)^{\wedge}$ and so by (4) $H\subseteq(H\cap(F\cup G))^{\wedge}$. We then have $H^{\sim}=H\subseteq(H\cap(F\cup G))^{\wedge\sim}$ so that

$$\begin{aligned}E^{\wedge\sim}\cap(F\cup G)^{\wedge\sim}&=H\subseteq(E^{\wedge\sim}\cap(F\cup G)^{\wedge\sim}\cap(F\cup G))^{\wedge\sim}=(E^{\wedge\sim}\cap(F\cup G))^{\wedge\sim}\\&=(E\cap(F\cup G))^{\wedge\sim}.\end{aligned}$$

The opposite inclusion is always valid and hence (3).

DEFINITION 5.1. For any E in S, let $E^{+}=X-\mathbf{U}$-closure (E), and let

$$\mathbf{S}^{+}=\{E^{+}\mid E\text{ in }\mathbf{S}\}.$$

THEOREM 5.1. *The lattice* $\mathbf{S}$ *is distributive if and only if for every* E *in* $\mathbf{S}$, $E\cup E^{+}$ *is dense in* X*, relative to the topology* $\mathbf{T}$ *generated by* $\mathbf{U}$ *and* $\mathbf{V}$.

PROOF. Suppose there is an E in $\mathbf{S}$ for which $E\cup E^{+}$ is not $\mathbf{T}$-dense. Select an $\bar{x}$ in X-$\mathbf{T}$-cl$(E\cup E^{+})$. Then, by the pairwise regularity of $(X,\mathbf{U},\mathbf{V})$ and the definition of $\mathbf{T}$, we can find sets V in $\mathbf{S}$ and U in $\mathbf{S}^{+}$ with $\bar{x}\in V\cap U$, and $V\cap U\subseteq\mathbf{U}$-cl$(E)-E$. Let $Q=\mathbf{V}$-int$(X-U)$. Then Q is in $\mathbf{S}$. We show that $V\subseteq\mathbf{U}$-cl$(Q\cup E)$.

If $x\in V$ then either $x\in U$ or $x\notin U$. If $x\in U\cap V$ then $x\in\mathbf{U}$-cl$(E)\subseteq\mathbf{U}$-cl$(Q\cup E)$. If $x\notin U$ but $x\in V$ then $x\in(X-U)\cap V$. Since $X-U$ is the $\mathbf{U}$-cl of its $\mathbf{V}$-int, we have $X-U=\mathbf{U}$-cl Q. Therefore $x\in\mathbf{U}$-cl $Q\subseteq\mathbf{U}$-cl$(Q\cup E)$. We now show that $\mathbf{S}$ is not distributive by showing that (4) of Proposition 5.1 fails.

If (4) were true, then we would have

$$V^{\wedge}=(V\cap(Q\cup E))^{\wedge}=((V\cap Q)\cup(V\cap E))^{\wedge}=(V\cap Q)^{\wedge}\cup(V\cap E)^{\wedge}.$$

Then, since $\bar{x}$ is in $V^{\wedge}$, we must have $\bar{x}$ in $(V\cap Q)^{\wedge}$ or in $(V\cap E)^{\wedge}$. If $\bar{x}\in(V\cap Q)^{\wedge}$ then $\bar{x}\in Q^{\wedge}=X-U$ and so $\bar{x}$ is not in U, which is false. If, on the other hand, we have $\bar{x}\in(V\cap E)^{\wedge}$, then U, which is a $\mathbf{U}$-neighborhood of x, must intersect $V\cap E$. This forces $U\cap V$ to intersect E, which it does not do. So (4) fails, and $\mathbf{S}$ is not distributive.

Now we suppose that **S** is not distributive, and show that there is an E in **S** with $E \cup E^+$ not **T**-dense. There are two cases to consider (Birkhoff (1967)), but the proof is the same in each one and so we consider only the case in which **S** contains five elements P, Q, R, S, T, with $Q \nsubseteq R$, $R \vee S = Q \vee S = T$, and $R \wedge S = Q \wedge S = P$ (the other case does not have $Q \nsubseteq R$; rather $R \wedge Q = P$, $R \vee Q = T$). We shall prove that the set $R \cap (X - Q^\wedge)$, which is in **T**, is non-empty and contained in $S^\wedge - S$. Since Q and R are in **S**, $Q^\wedge \nsubseteq R^\wedge$, and $R \nsubseteq Q^\wedge$, so $R - Q^\wedge \neq \varnothing$. We show $R - Q^\wedge \subseteq S^\wedge - S$. We know that $R^\wedge \cup S^\wedge = Q^\wedge \cup S^\wedge$ so that

$$(R^\wedge \cup S^\wedge) \cap (X - S^\wedge) = (Q^\wedge \cup S^\wedge) \cap (X - S^\wedge),$$

and so $R^\wedge \cap (X - S^\wedge) = Q^\wedge \cap (X - S^\wedge)$. Therefore, $R \cap (X - Q^\wedge) \subseteq S^\wedge$. But if x is a point of $R \cap (X - Q^\wedge)$ and also of S, then $x \in R \cap S = Q \cap S$. So $x \in Q$, a contradiction. Letting $E = S$, it is clear that the complement of $E \cup E^+$ contains the non-empty **T**-set $R - Q^\wedge$. The theorem follows.

COROLLARY 5.1. *If* $\mathbf{U} \subseteq \mathbf{V}$ *or* $\mathbf{V} \subseteq \mathbf{U}$ *then* **S** *is distributive.*

PROOF. Suppose, for instance, that $\mathbf{U} \subseteq \mathbf{V}$. Then $\mathbf{T} = \mathbf{V}$ and if E is in **S**, then E is **V**-open. Therefore E is (**V**) regular open and E^+ is its complement in the Boolean algebra of (**V**) regular open sets. By a standard argument (see, for example, Halmos (1963), p. 15) we conclude that $E \cup E^+$ is **V**-dense.

In his 1944 article, Funayama gives an example of a distributive lattice L whose normal completion, L', is not even modular. He also gives a necessary and sufficient condition on L for L' to be distributive. We use Priestley's representation for L, along with the theorem just presented, to obtain Funayama's condition, in somewhat altered form.

According to Priestley's theorem, there is a bitopological space $(Y, \mathbf{D}, \mathbf{I})$, as discussed earlier, such that L is isomorphic to the lattice of **I**-open, **D**-closed subsets, **L**, via the association of l with $\{y \mid y(l) = 1\}$. The minimal completion of **L** is the lattice of subsets E of Y with $E = \mathbf{I}\text{-int}(\mathbf{D}\text{-cl}(E))$, which we denote by **K**. We may then identify **K** with L'.

REMARK. We use **K**, instead of **S** to avoid confusion. However, the results just given for **S** apply equally to **K**.

As we saw earlier, E is in **K** if and only if $E = \{y \mid y(l) = 1 \text{ for some } l \in M\}$ where M is a normal subset of L. Then the set E^+, which is Y-**D**-cl(E), has the form $E^+ = \{y \mid y(l) = 0 \text{ for some } l \text{ in } M^*\}$. If L' (and hence **K**) is not distributive, then there is some E in **K** such that $E \cup E^+$ is not **T**-dense in Y, where **T** is the topology generated by **I** and **D**. Therefore there is a **T** set, T, of the form

$$T = \{y \mid y(l_0) = 0 \text{ and } y(l_1) = 1\},$$

contained in $Y-(E \cup E^+)$. Therefore, whenever $y(l_0) = 0$ and $y(l_1) = 1$, it is the case that y is identically 0 on M and identically 1 on M^*.

Conversely, suppose that within the distributive lattice L we can find a normal subset M and two elements l_0 and l_1, with $l_1 \nleqslant l_0$, such that for every y in Y for which $y(l_0) = 0$ and $y(l_1) = 1$, y is identically 0 on M and identically 1 on M^*. It follows then that $l_1 \wedge m \leqslant l_0$ for all m in M, while $l_0 \vee n \geqslant l_1$ for all n in M^*. Within the lattice of normal subsets, then, we have

$$M \vee [l_0] = M \vee [l_1 \vee l_0] \quad \text{and} \quad M \wedge [l_0] = M \wedge [l_1 \vee l_0], \quad \text{with } [l_1 \vee l_0] \neq [l_0].$$

The sub-lattice consisting of the elements M, $[l_0]$, $[l_1 \vee l_0]$, $M \vee [l_1 \vee l_0] = M \vee [l_0]$ and $M \wedge [l_1 \vee l_0] = M \wedge [l_0]$ is then non-distributive. In fact, it is even non-modular.

We can summarize these findings as follows:

THEOREM 5.2. *If L is a distributive lattice and L' its normal completion, then L' is non-distributive if and only if there is a normal subset M in L, and distinct elements l_0 and l_1 in L, such that $l_0 \vee n \geqslant l_1$ for all n in M^*, and $l_1 \wedge m \leqslant l_0$ for all m in M. In this case, L' is also non-modular.*

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