Notes on Bessel's Equation and the Gamma Function

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1 Bessel's Equations

For each non-negative constant p, the associated *Bessel Equation* is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0, \tag{1.1}$$

which can also be written in the form

$$y'' + P(x)y' + Q(x)y = 0, \tag{1.2}$$

with $P(x) = \frac{1}{x}$ and $Q(x) = 1 - \frac{p^2}{x^2}$.

Solutions of Equation (1.1) are *Bessel functions.* These functions first arose in Daniel Bernoulli's study of the oscillations of a hanging chain, and now play important roles in many areas of applied mathematics [1].

We begin this note with Bernoulli's problem, to see how Bessel's Equation becomes involved. We then consider Frobenius-series solutions to second-order linear differential equations with regular singular points; Bessel's Equation is one of these. Once we obtain the Frobenius-series solution of Equation (1.1), we discover that it involves terms of the form $p!$, for (possibly) non-integer p. This leads to the *Gamma Function*, which extends the factorial function to such non-integer arguments.

The Gamma Function, defined for $x > 0$ by the integral

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \tag{1.3}$$

is a *higher transcendental* function that cannot be evaluated by purely algebraic means, and can only be approximated by numerical techniques. With clever changes of variable, a large number of challenging integration problems can be rewritten and solved in terms of the gamma function.

We prepare for our discussion of Bernoulli's hanging chain problem by recalling some important points in the derivation of the one-dimensional wave equation for the vibrating string problem.

2 The Vibrating String Problem

In the vibrating string problem, the string is fixed at end-points $(0,0)$ and $(1,0)$. The position of the string at time t is given by $y(x,t)$, where x is the horizontal spatial variable. It is assumed that the string has a constant mass density, m. Consider the small piece of the string corresponding to the interval $[x, x+\Delta x]$. Its mass is $m\Delta x$, and so, from Newton's equating of force with mass times acceleration, we have that the force f on the small piece of string is related to acceleration by

$$f = m(\Delta x)\frac{\partial^2 y}{\partial t^2}. \tag{2.1}$$

In this problem, the force is not gravitational, but comes from the tension applied to the string; we denote by $T(x)$ the tension in the string at x. This tensile force acts along the tangent to the string at every point. Therefore, the force acting on the left end-point of the small piece is directed to the left and is given by $-T(x)\sin(\theta(x))$; at the right end-point it is $T(x+\Delta x)\sin(\theta(x+\Delta x))$, where $\theta(x)$ is the angle the tangent line at x makes with the horizontal. For small-amplitude oscillations of the string, the angles are near zero and the sine can be replaced by the tangent. Since $\tan(\theta(x)) = \frac{\partial y}{\partial x}(x)$, we can write the net force on the small piece of string as

$$f = T(x+\Delta x)\frac{\partial y}{\partial x}(x+\Delta x) - T(x)\frac{\partial y}{\partial x}(x). \tag{2.2}$$

Equating the two expressions for f in Equations (2.1) and (2.2) and dividing by Δx, we obtain

$$\frac{T(x+\Delta x)\frac{\partial y}{\partial x}(x+\Delta x) - T(x)\frac{\partial y}{\partial x}(x)}{\Delta x} = m\frac{\partial^2 y}{\partial t^2}. \tag{2.3}$$

Taking limits, as $\Delta x \to 0$, we arrive at the *Wave Equation*

$$\frac{\partial}{\partial x}\Big(T(x)\frac{\partial y}{\partial x}(x)\Big) = m\frac{\partial^2 y}{\partial t^2}. \tag{2.4}$$

For the vibrating string problem, we also assume that the tension function is constant, that is, $T(x) = T$, for all x. Then we can write Equation (2.4) as the more familiar

$$T\frac{\partial^2 y}{\partial x^2} = m\frac{\partial^2 y}{\partial t^2}. \tag{2.5}$$

We could have introduced the assumption of constant tension earlier in this discussion, but we shall need the wave equation for variable tension Equation (2.4) when we consider the hanging chain problem.

3 The Hanging Chain Problem

Imagine a flexible chain hanging vertically. Assume that the chain has a constant mass density m. Let the origin $(0,0)$ be the bottom of the chain, with the positive x-axis running vertically, up through the chain. The positive y-axis extends horizontally to the left, from the bottom of the chain. As before, the function $y(x,t)$ denotes the position of each point on the chain at time t. We are interested in the oscillation of the hanging chain. This is the vibrating string problem turned on its side, except that now the tension is not constant.

3.1 The Wave Equation for the Hanging Chain

The tension at the point x along the chain is due to the weight of the portion of the chain below the point x, which is then $T(x) = mgx$. Applying Equation (2.4), we have

$$\frac{\partial}{\partial x}\Big(mgx\frac{\partial y}{\partial x}(x)\Big) = m\frac{\partial^2 y}{\partial t^2}. \tag{3.1}$$

As we normally do at this stage, we separate the variables, to find potential solutions.

3.2 Separating the Variables

We consider possible solutions having the form

$$y(x,t) = u(x)v(t). \tag{3.2}$$

Inserting this $y(x,t)$ into Equation (3.1), and doing a bit of algebra, we arrive at

$$gxu''(x) + gu'(x) + \lambda u(x) = 0, \tag{3.3}$$

and

$$v''(t) + \lambda v(t) = 0, \tag{3.4}$$

where λ is the separation constant. It is Equation (3.3), which can also be written as

$$\frac{d}{dx}(gxu'(x)) + \lambda u(x) = 0, \tag{3.5}$$

that interests us here.

3.3 Obtaining Bessel's Equation

With a bit more work, using the change of variable $z = 2\sqrt{\frac{\lambda}{g}}\sqrt{x}$ and the Chain Rule (no pun intended!), we find that we can rewrite Equation (3.3) as

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - 0^2)u = 0, \tag{3.6}$$

which is Bessel's Equation (1.1), with the parameter value $p = 0$.

4 Solving Bessel's Equations

Second-order linear differential equations with the form

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \tag{4.1}$$

with neither $P(x)$ nor $Q(x)$ analytic at $x = x_0$, but with both $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ analytic, are said to be equations with *regular singular points*. Writing Equation (1.1) as

$$y''(x) + \frac{1}{x}y'(x) + (1 - \frac{p^2}{x^2})y(x) = 0, \tag{4.2}$$

we see that Bessel's Equation is such a regular singular point equation, with the singular point $x_0 = 0$. Solutions to such equations can be found using the technique of Frobenius series.

4.1 Frobenius-series solutions

A Frobenius series associated with the singular point $x_0 = 0$ has the form

$$y(x) = x^m(a_0 + a_1x + a_2x^2 + ...), \tag{4.3}$$

where m is to be determined, and $a_0 \neq 0$. Since $xP(x)$ and $x^2Q(x)$ are analytic, we can write

$$xP(x) = p_0 + p_1x + p_2x^2 + ..., \tag{4.4}$$

and

$$x^2Q(x) = q_0 + q_1x + q_2x^2 + ..., \tag{4.5}$$

with convergence for $|x| < R$. Inserting these expressions into the differential equation, and performing a bit of algebra, we arrive at

$$\sum_{n=0}^{\infty} \left\{ a_n[(m+n)(m+n-1) + (m+n)p_0 + q_0] + \sum_{k=0}^{n-1} a_k[(m+k)p_{n-k} + q_{n-k}] \right\} x^n = 0.$$

(4.6)

Setting each coefficient to zero, we obtain a recursive algorithm for finding the a_n. To start with, we have

$$a_0[m(m-1) + mp_0 + q_0] = 0. \tag{4.7}$$

Since $a_0 \neq 0$, we must have

$$m(m-1) + mp_0 + q_0 = 0; \tag{4.8}$$

this is called the *Indicial Equation*. We solve the quadratic Equation (4.8) for $m = m_1$ and $m = m_2$.

4.2 Bessel Functions

Applying these results to Bessel's Equation (1.1), we see that $P(x) = \frac{1}{x}$, $Q(x) = 1 - \frac{p^2}{x^2}$, and so $p_0 = 1$ and $q_0 = -p^2$. The Indicial Equation (4.8) is now

$$m^2 - p^2 = 0, \tag{4.9}$$

with solutions $m_1 = p$, and $m_2 = -p$. The recursive algorithm for finding the a_n is

$$a_n = -a_{n-2}/n(2p+n). \tag{4.10}$$

Since $a_0 \neq 0$ and $a_{-1} = 0$, it follows that the solution for $m = p$ is

$$y = a_0 x^p \Big[1 - \frac{x^2}{2^2(p+1)} + \frac{x^4}{2^4 2!(p+1)(p+2)} - ... \Big]. \tag{4.11}$$

Setting $a_0 = 1/2^p p!$, we get the pth Bessel function,

$$J_p(x) = \sum_{n=0}^{\infty} (-1)^n \Big(\frac{x}{2}\Big)^{2n+p} / n!(p+n)!. \tag{4.12}$$

The most important Bessel functions are $J_0(x)$ and $J_1(x)$.

We have a Problem! So far, we have allowed p to be any real number. What, then, do we mean by $p!$ and $(n+p)!$? To answer this question, we need to investigate the gamma function.

5 The Gamma Function

We want to define $p!$ for p not a non-negative integer. The Gamma Function is the way to do this.

5.1 Extending the Factorial Function

As we said earlier, the Gamma Function is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \tag{5.1}$$

Using integration by parts, it is easy to show that

$$\Gamma(x+1) = x\Gamma(x). \tag{5.2}$$

Using Equation (5.2) and the fact that

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1, \tag{5.3}$$

we obtain

$$\Gamma(n+1) = n!, \tag{5.4}$$

for $n = 0, 1, 2, ...$

5.2 Extending $\Gamma(x)$ to negative x

We can use

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \tag{5.5}$$

to extend $\Gamma(x)$ to any $x < 0$, with the exception of the non-negative integers, at which $\Gamma(x)$ is unbounded.

5.3 An Example

We have

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-1/2} dt. \tag{5.6}$$

Therefore, using $t = u^2$, we have

$$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-u^2} du. \tag{5.7}$$

Squaring, we get

$$\Gamma(\frac{1}{2})^2 = 4\int_0^\infty \int_0^\infty e^{-u^2} e^{-v^2} dudv. \tag{5.8}$$

In polar coordinates, this becomes

$$\Gamma(\frac{1}{2})^2 = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2} r dr d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1 d\theta = \pi. \tag{5.9}$$

Consequently, we have

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}. \tag{5.10}$$

6 An Application of the Bessel Functions in Astronomy

In remote sensing applications, it is often the case that what we measure is the Fourier transform of what we really want. This is the case in medical imaging, for example, in both x-ray tomography and magnetic-resonance imaging. It is also often the case in astronomy. Consider the problem of determining the size of a distant star.

We model the star as a distance disk of uniform brightness. Viewed as a function of two variables, it is the function that, in polar coordinates, can be written as $f(r, \theta) = g(r)$, that is, it is a radial function that is a function of r only, and independent of θ. The function $g(r)$ is, say, one for $0 \leq r \leq R$, where R is the radius of the star, and zero, otherwise. From the theory of Fourier transform pairs in two-dimensions, we know that the two-dimensional Fourier transform of f is also a radial function; it is the function

$$H(\rho) = 2\pi\int_0^R r J_0(r\rho) dr,$$

where J_0 is the zero-th order Bessel function. From the theory of Bessel functions, we learn that

$$\frac{d}{dx}[xJ_1(x)] = xJ_0(x),$$

so that

$$H(\rho) = \frac{2\pi}{\rho} R J_1(R\rho).$$

When the star is viewed through a telescope, the image is blurred by the atmosphere. It is commonly assumed that the atmosphere performs a convolution filtering on the light from the star, and that this filter is random and varies somewhat from one observation to another. Therefore, at each observation, it is not $H(\rho)$, but $H(\rho)G(\rho)$ that is measured, where $G(\rho)$ is the filter transfer function operating at that particular time.

Suppose we observe the star N times, for each $n = 1, 2, ..., N$ measuring values of the function $H(\rho)G_n(\rho)$. If we then average over the various measurements, we can safely say that the first zero we observe in our measurements is the first zero of $H(\rho)$, that is, the first zero of $J_1(R\rho)$. The first zero of $J_1(x)$ is known to be about 3.8317, so knowing this, we can determine R. Actually, it is not truly R that we are measuring, since we also need to involve the distance D to the star, known by other means. What we are measuring is the perceived radius, in other words, half the subtended angle. Combining this with our knowledge of D, we get R.

7 Orthogonality of Bessel Functions

As we have seen previously, the orthogonality of trigonometric functions plays an important role in Fourier series. A similar notion of orthogonality holds for Bessel functions. We begin with the following theorem.

Theorem 7.1 *Let $u(x)$ be a non-trivial solution of $u''(x) + q(x)u(x) = 0$. If*

$$\int_1^\infty q(x)dx = \infty \,,$$

then $u(x)$ has infinitely many zeros on the positive x-axis.

Bessel's Equation

$$x^2y''(x) + xy'(x) + (x^2 - p^2)y(x) = 0, \tag{7.1}$$

can be written in *normal form* as

$$y''(x) + \left(1 + \frac{1 - 4p^2}{4x^2}\right)y(x) = 0, \tag{7.2}$$

and, as $x \to \infty$,

$$q(x) = 1 + \frac{1 - 4p^2}{4x^2} \to 1,$$

so, according to the theorem, every non-trivial solution of Bessel's Equation has infinitely many positive zeros.

Now consider the following theorem, which is a consequence of the Sturm Comparison Theorem to be discussed later.

Theorem 7.2 *Let $y_p(x)$ be a non-trivial solution of Bessel's Equation*

$$x^2y''(x) + xy'(x) + (x^2 - p^2)y(x) = 0,$$

for $x > 0$. If $0 \leq p < \frac{1}{2}$, then every interval of length π contains at least one zero of $y_p(x)$; if $p = \frac{1}{2}$, then the distance between successive zeros of $y_p(x)$ is precisely π; and if $p > \frac{1}{2}$, then every interval of length π contains at most one zero of $y_p(x)$.

It follows from these two theorems that, for each fixed p, the function $y_p(x)$ has an infinite number of positive zeros, say $\lambda_1 < \lambda_2 < ...$, with $\lambda_n \to \infty$.

For fixed p, let $y_n(x) = y_p(\lambda_n x)$. We have the following orthogonality theorem.

Theorem 7.3 *For $m \neq n$, $\int_0^1 x y_m(x) y_n(x) dx = 0$.*

Proof: Let $u(x) = y_m(x)$ and $v(x) = y_n(x)$. Then we have

$$u'' + \frac{1}{x}u' + (\lambda_m^2 - \frac{p^2}{x^2})u = 0,$$

and

$$v'' + \frac{1}{x}v' + (\lambda_n^2 - \frac{p^2}{x^2})v = 0.$$

Multiplying on both sides by x and subtracting one equation from the other, we get

$$x(uv'' - vu'') + (uv' - vu') = (\lambda_m^2 - \lambda_n^2)xuv.$$

Since

$$\frac{d}{dx}\Big(x(uv' - vu')\Big) = x(uv'' - vu'') + (uv' - vu'),$$

it follows, by integrating both sides over the interval $[0, 1]$, that

$$x(uv' - vu')|_0^1 = (\lambda_m^2 - \lambda_n^2)\int_0^1 xu(x)v(x)dx.$$

But

$$x(uv' - vu')|_0^1 = u(1)v'(1) - v(1)u'(1) = 0.$$

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References

[1] Simmons, G. (1972) *Differential Equations, with Applications and Historical Notes*. New York: McGraw-Hill.