JOURNAL OF MULTIVARIATE ANALYSIS 2, 1–13 (1972)

Contractive Projections with Contractive Complement in L_p Space

CHARLES BYRNE, JR.

Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15213

AND

FRANCIS E. SULLIVAN

Department of Computer Science, University of Pittsburgh, Pittsburgh, Pennsylvania 15213

Communicated by M. M. Rao

Using the concepts of conditional expectation and independence of sub-algebras, we characterize those contractive projections, P, on L_p, over a probability measure space, having the property that $I - P$ is contractive. By contractive projection we mean a linear operator, P, on the Lebesgue space, L_p, $1 < p < \infty$, $\neq 2$, with $P^2 = P$, $\| P \| = 1$.

INTRODUCTION

If L_p, $1 < p < \infty$, $\neq 2$, is a Lebesgue space on a probability measure space (X, Σ, m), and $P : L_p \to L_p$ is a contractive projection (a linear operator with $P^2 = P$, $\| P \| = 1$) such that $P(1) = 1$, then P is a conditional expectation, $E_{\mathscr{B}}$, relative to some sub-σ-algebra, $\mathscr{B}$, of Σ. Ando (c.f. [1]) shows that every contractive projection, P, on L_p induces a conditional expectation, in a natural way, and hence the concept of conditional expectation is central to our discussion. In this paper we consider contractive projections, P, on L_p, having the additional property that the complement of P, the projection, $I - P$, is contractive. If U is a linear isometry on L_p such that $U^2 = I$, then the projections $P = (I + U)/2$

Received September 30, 1971; revised November 15, 1971.

AMS 1970 subject classification: Primary 46E30, 62P99; Secondary 41A50, 28A65, 28A60.

Key words and phrases: Conditional expectation; contractive projection; cycle subspaces of L_p; independence of sub- σ-algebras; regular set isomorphisms; mean ergodic theorem; martingale.

and $I - P = (I - U)/2$ are of this type. The main result in this paper is the converse of this observation; P and $I - P$ are both contractive iff (if and only if) $P = (I + U)/2$ for some isometry, U, with $U^2 = I$ (such U will be called "reflections"). The analysis depends heavily on the concept and properties of independent sub-σ-algebras, in the sense of probability theory (c.f. [6]).

In the first section we discuss the results of Lamperti and Ando, (c.f. [5, 1]), concerning isometries and contractive projections on L_p, respectively. In the following section we develop properties of reflections and prove the main result. We conclude with a discussion of general isometries and show how every isometry generates a reflection in a natural way.

Notation. If E is a set in Σ, then we denote by ϕ_E the characteristic function of the set E, and by χ_E the characteristic projection, $\chi_E(f) = f \cdot \phi_E$. The complement of the set E is denoted by E'. For $f \in L_p$, we denote by $\bar{f}^{p-1}$ that function in L_q with the property $|f|^p = f \cdot \bar{f}^{p-1}$ (where $p^{-1} + q^{-1} = 1$). By $N(f)$ we mean the support of f.

1. Isometries and Contractive Projections

If $U : L_p \to L_p$ is an isometry, then U induces a set mapping, $T : \Sigma \to \Sigma$, defined by $T(E) = N(U(\phi_E))$. If we then define a set function, m^*, on the range of T, by $m^*(T(A)) = m(A)$, it can be shown that m^* is a measure, absolutely continuous, with respect to m (restricted to the range of T). Let $|h|^p$ be the Radon–Nikodym derivative of m^*, with respect to m. Then, we can describe U by the formula

$$U(\phi_E) = h \cdot \phi_{T(E)}, \qquad \text{for} \quad E \in \Sigma. \tag{1}$$

The set mapping, T, is a regular set isomorphism; i.e.,

(a) $T(X - E) = T(X) - T(E)$;
(b) $T(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} T(A_n)$, for disjoint A_n ;
(c) $m(T(E)) = 0$ iff $m(E) = 0$.

It can be shown that these three properties imply the following properties:

(d) $T(A) \subseteq T(B)$ iff $A \subseteq B$;
(e) $T(A) \cap T(B) = \varnothing$ iff $A \cap B = \varnothing$;
(f) $T(A - B) = T(A) - T(B)$;
(g) $T(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} T(A_n)$, for all A_n ;
(h) $T(\bigcap_{n=1}^{k} A_n) = \bigcap_{n=1}^{k} T(A_n)$.

The range of T is a sub-σ-algebra of Σ. In general, we cannot say that $T(X) = X$. However, if U is a reflection, then the induced set mapping, T, has the property that $T(T(E)) = E$, and so, by (d), $T(X) = X$. In what follows, we shall assume that $T(X) = X$. We shall study reflections by studying the induced set mapping. It is easy to see that every set mapping having the properties (a), (b), and (c) induces an isometry, in the manner of (1). For proofs and details, see Lamperti [5].

The contractive projections, P, on L_p having the property

$$\|f\|_p^p = \|P(f)\|_p^p + \|(I - P)(f)\|_p^p \tag{2}$$

have been characterized; they are characteristic projections (c.f. [7]). If U is a reflection on L_p, then, for $E \in \Sigma$, the mapping $U \circ \chi_E \circ U^{-1}$ is a contractive projection on L_p, with property (2). Therefore, there is a set, $H(E)$, such that $\chi_{H(E)} = U \circ \chi_E \circ U^{-1}$. We shall show that $H = T$, where T is the regular set isomorphism associated with U.

Remark. If $f \in L_p$, and $\mathscr{B}$ is a sub-σ-algebra of Σ, then $S(f, \mathscr{B})$ is the closed linear subspace of L_p spanned by elements of the form $f \cdot \phi_E$, $E \in \mathscr{B}$. These are the cycle subspaces, and we have

$$S(f, \Sigma) = S(\phi_{N(f)}, \Sigma),$$

and

$$S(f, \mathscr{B}) = S(\phi_{N(f)}, \mathscr{B})$$

iff f is $\mathscr{B}$-measurable.

LEMMA 1. *For any $f \in L_p$, and reflection, U, we have $U[S(f, \Sigma)] = S(Uf, \Sigma)$.*

Proof. It follows from the definition of H that, for any set $E \in \Sigma$, $U(\phi_E f) = \phi_{H(E)} \cdot Uf$, and, consequently, $U[S(f, \Sigma)] \subseteq S(Uf, \Sigma)$. Since U^{-1} is also a reflection, we can prove, similarly, that $U^{-1}[S(Uf, \Sigma)] \subseteq S(f, \Sigma)$.

COROLLARY 1. *For each $E \in \Sigma$, $T(E) = H(E)$.*

Proof. From Lemma 1 and the definitions of H and T, we have

$$\begin{aligned} \chi_{H(E)}[L_p] &= U \circ \chi_E \circ U^{-1}[L_p] \\ &= U \circ \chi_E[L_p] \\ &= U \circ \chi_E[S(1, \Sigma)] \\ &= U[S(\phi_E, \Sigma)] \\ &= S(U\phi_E, \Sigma) \\ &= S(\phi_{T(E)}, \Sigma) \\ &= \chi_{T(E)}[L_p]. \end{aligned}$$

COROLLARY 2. *There is a set E ($\neq \varnothing$, $\neq X$) such that $H(E) = E$, iff there is $f \in L_p$, $f \neq 0$, with $N(f) \neq X$, and $N(f) = N(Uf)$.*

Proof. If such an f exists, then, using the same arguments as in Corollary 1, we show that $H(N(f)) = N(f)$. Conversely, if such an E exists, $f = \phi_E$ has the desired properties.

If P is a contractive projection on L_p, and $P(1) = 1$, then P is a conditional expectation, $E_{\mathscr{B}}$, for some sub-σ-algebra, $\mathscr{B}$, of Σ (c.f. [1]). Even if $P(1) \neq 1$, there is a function, f, in the range of P, such that $N(g) \subseteq N(f)$ for all g in the range of P. Let f be such a function, and define the Lebesgue space, Y_f, by

$$Y_f = L_p(N(f), \Sigma \cap N(f), |f|^p m),$$

where, by $|f|^p m$ we mean the measure $(|f|^p m)(A) = \int_A |f|^p \, dm$. Now, define a mapping, P^*, on Y_f, by

$$P^*(k) = P(f \cdot k)/f,$$

for $k \in Y_f$. It is easily shown that P^* is a contractive projection on Y_f, and that $P^*(1) = 1$ (where, of course, "1" means "$\phi_{N(f)}$"). It then follows that P^* is a conditional expectation, relative to some sub-σ-algebra, $\mathscr{B}^*$, of $\Sigma \cap N(f)$ and the measure, $|f|^p m$. The class $\mathscr{B}^*$, considered as a sub-class of Σ, is not an algebra, but is a sub-σ-ring with maximum element, $N(f)$. Let $\mathscr{B}$ be the unique sub-σ-algebra of Σ containing $\mathscr{B}^*$, and having the properties

(a) $E \cap N(f) \in \mathscr{B}$ iff $E \cap N(f) \in \mathscr{B}^*$, for all $E \in \Sigma$;

(b) if $E \cap N(f) = \varnothing$, then $E \in \mathscr{B}$, for all $E \in \Sigma$.

We shall denote the conditional expectation, P^*, by $P^* = E_{\mathscr{B}}'$. The context will clarify any possible ambiguity. We can see, now, that if P is a contractive projection on L_p, then there are $f \in L_p$, and $\mathscr{B}$, a sub-σ-algebra of Σ, such that

$$P(g) = f E_{\mathscr{B}}'(g/f).$$

Therefore, the range of P is $S(f, \mathscr{B})$. Since L_p is smooth, P is determined by its range (c.f. [3]).

We shall have need, later, of the concept of independence of two sub-σ-algebras. We say that $\mathscr{A}$ and $\mathscr{B}$ are independent if, for every $A \in \mathscr{A}$, $B \in \mathscr{B}$, we have $m(A \cap B) = m(A) \cdot m(B)$. For any measurable function f, let $\mathscr{B}(f)$ be the sub-σ-algebra of Σ generated by sets of the form $f^{-1}(A)$, for Borel sets A, in the complex plane. We say that two measurable functions, f and g, are independent (f and $\mathcal{O}\!\ell$ are independent) iff $\mathscr{B}(f)$ and $\mathscr{B}(g)$ are ($\mathscr{B}(f)$ and $\mathcal{O}\!\ell$ are). If f and g are in L_1 and are independent, then $f \cdot g$ is in L_1, and

$$\int f \cdot g \, dm = \int f \, dm \cdot \int g \, dm.$$

See Loève [6] for details.

2. Contractive Complementary Pairs

A simple example will illustrate the concepts involved in this discussion and may help to motivate some of the definitions. Consider the space, l_p^4, of four-tuples, with the usual l_p norm. The mapping $U(a, b, c, d) = (b, a, d, c)$ is a reflection on this space, and its invariant subspace consists of four-tuples of the form (a, a, b, b). Let $X = \{1, 2, 3, 4\}$, and $\Sigma = 2^X$. Then, if $\mathcal{O}\!\ell = \{\varnothing, \{1, 2\}, \{3, 4\}, X\}$, we can see that the invariant subspace of U is $S(1, \mathcal{O}\!\ell)$. The projection, $P = (I + U)/2$, is $E_{\mathcal{O}\!\ell}$. The projection $I - P$ has range $S(g, \mathcal{O}\!\ell)$, where $g = (1, -1, 1, -1)$. We find, also, that $\mathscr{B}(g) = \{\varnothing, \{1, 3\}, \{2, 4\}, X\}$, and hence $\mathscr{B}(g)$ and $\mathcal{O}\!\ell$ are independent. Returning to the general L_p space considered above, we make

Definition 1. A pair of complementary contractive projections $\{P, I - P\}$, with ranges $S(f, \mathcal{O}\!\ell)$ and $S(g, \mathcal{O}\!\ell)$, respectively, is said to be total if $N(f) = N(g) = X$. A total pair is called independent if, for some choice of f and g in the representation of the ranges, it is the case that $\mathcal{O}\!\ell$ and g/f are independent (for the measure $|f|^p m$). As we shall see, this implies independence of $\mathcal{O}\!\ell$ and f/g for the measure $|g|^p m$.

Remark. If contractive projection, P, has range $S(f, \mathscr{B})$, then $E \in \mathscr{B}$ iff χ_E and P commute. Therefore, in the case of the total pair $\{P, I - P\}$, the same algebra will generate each range.

What we discovered, in the example above, is that the reflection, U, gave rise to an independent pair $\{P, I - P\}$. With some slight restriction on U, this is always the case, and this result will be fundamental to the proof of the main theorem.

Definition 2. Let U be an isometry on L_p. We say that U is reduced, if $U\chi_E = \chi_E$ implies $E = \varnothing$.

Remarks. The isometry, U, is reduced iff the invariant subspace does not contain any cycle of the form $S(\phi_E, \Sigma)$. An isometry, U, is reduced, iff its associated regular set isomorphism has the property

$$\text{"for any } A \in \Sigma, \exists F \in \Sigma, F \subseteq A, \text{ with } T(F) \neq F\text{"}.$$

For an isometry, U, let E be the largest set in Σ such that $U\chi_E = \chi_E$. Then $U\chi_{E'}$ is a reduced isometry on the L_p space $S(\phi_{E'}, \Sigma)$. Therefore, any isometry can be considered as a reduced isometry, restricted to a subspace $S(\phi_{E'}, \Sigma)$, and the identity, restricted to $S(\phi_E, \Sigma)$.

The first theorem we shall prove is

THEOREM 1. *The following are equivalent:*

(a) *There is a reduced reflection, U, on L_p, with invariant subspace, M;*

(b) *There is an independent pair of contractive projections on L_p, with range, M, and null manifold, M, respectively.*

(c) *There is a sub-σ-algebra, $\mathcal{A}$, of Σ, and there is a set B, $\neq \varnothing$, $\neq X$, $B \in \Sigma$, such that, for every $E \in \Sigma$,*

$$E = (A \cap B) \cup (C \cap B'),$$

where A and C are in $\mathcal{A}$.

We shall prove this theorem by examining the induced regular set isomorphism associated with a reflection.

Let U be a reflection on L_p. Then, the associated T has the property $T(T(E)) = E$ for all $E \in \Sigma$. Let

$$\mathcal{A} = \{A \in \Sigma \mid A = T(A)\} \quad \text{and} \quad \mathcal{K} = \{K \in \Sigma \mid K \cap T(K) = \varnothing\}.$$

Then $\mathcal{A}$ is a sub-σ-algebra of Σ. A set B in $\mathcal{K}$ is said to be maximal in $\mathcal{K}$ if $B \subset C$, $B \neq C$, implies $C \notin \mathcal{K}$.

LEMMA 2. *$\mathcal{K}$ has a maximal element.*

Proof. We shall show that every increasing chain in $\mathcal{K}$ has an upper bound, in $\mathcal{K}$. We may assume that any such chain is, at most, countable. Suppose, then, that $B_1 \subseteq B_2 \subseteq \cdots$ is such a chain in $\mathcal{K}$. Then, if $D = \bigcup_{n=1}^{\infty} B_n$, we show that $D \in \mathcal{K}$. Since B_n is in $\mathcal{K}$ for all n, we have, for all n,

$$\phi_{B_n} + \phi_{T(B_n)} = (\phi_{B_n} + \phi_{T(B_n)})^2.$$

The left side converges, in norm, to $\phi_D + \phi_{T(D)}$, while the right side converges to its square. It follows that $D \cap T(D) = \varnothing$. So, $D \in \mathscr{K}$. Applying Zorn's lemma, we have the assertion.

LEMMA 3. *Every set, E, in Σ, is the disjoint union of a set from $\mathfrak{A}$ and a set from $\mathscr{K}$.*

Proof. Write $E = [E \cap T(E)] \cup [E - T(E)]$.

LEMMA 4. *The reflection, U, is reduced iff, for all B, maximal in $\mathscr{K}$, $T(B) = B'$.*

Proof. Suppose U is reduced, and consider the set $B' - T(B)$. By Lemma 3 there are $A \in \mathfrak{A}$ and $K \in \mathscr{K}$ such that $B' - T(B) = A \cup K$. If $A \neq \varnothing$, then there is $D \subsetneq A$ such that $T(D) \neq D$. Let $C = B \cup [D - T(D)]$. Then $C \in \mathscr{K}$, contradicting the maximality of B. So $A = \varnothing$, and $B' - T(B) \in \mathscr{K}$. But, now $B \cup [B' - T(B)]$ is in $\mathscr{K}$, again contradicting the maximality of B. So $B' = T(B)$. Conversely, if $T(B) = B'$, then, given any set A, one of $A \cap B$, $A \cap B'$ is nonempty, and therefore is moved by T. It follows that U is reduced.

LEMMA 5. *Every set, E, in Σ, has the form $E = (A \cap B) \cup (C \cap B')$, for some A and C in $\mathfrak{A}$, and for B, maximal in $\mathscr{K}$.*

Proof. Let $E_1 = E \cap B$, $E_2 = E \cap B'$. Then let $A = E_1 \cup T(E_1)$, $C = E_2 \cup T(E_2)$.

LEMMA 6. *Let $\mathscr{B} = \{\varnothing, B, B', X\}$. If $U(1) = 1$, then $\mathscr{B}$ and $\mathfrak{A}$ are independent, relative to m.*

Proof. If $U(1) = 1$, then $h = 1$ (where h is the function associated with U). Therefore $m(T(E)) = m(E)$ for all $E \in \Sigma$. Suppose, for some $A \in \mathfrak{A}$, we have $m(A \cap B) < m(A) \cdot m(B)$. Since $m(A \cap B) = m(T(A \cap B)) = m(A \cap B')$, and $m(B) = m(B')$, we have $m(A \cap B') < m(A)\, m(B')$. But, then we obtain $m(A) < m(A)$. A similar argument works for $m(A \cap B) > m(A) \cdot m(B)$, and so $m(A \cap B) = m(A) \cdot m(B)$.

If $U(1) = 1$, then $(I + U)/2 = E_{\mathfrak{A}}$, since the range of $(I + U)/2$ is $S(1, \mathfrak{A})$. The complement, $I - E_{\mathfrak{A}}$, has range $S(g, \mathfrak{A})$, for $g = \phi_B - \phi_{B'}$. The above lemma tells us that this pair of contractive projections is independent.

If $U(1) = h$, $h \neq 1$, we let $f = 1 + h$, and consider the map, V, defined on the L_p space Y_f by

$$V(k) = U(f \cdot k)/f.$$

Clearly V is a reduced reflection, and $V(1) = 1$. Applying the above discussion to V, we see that $\{(I + V)/2, (I - V)/2\}$ is an independent pair of contractive projections, for the measure $|f|^p m$. The contractive projections $(I + U)/2$ and $(I - U)/2$ can be shown to be independent, for the measure m: let the ranges of $(I + V)/2$ and $(I - V)/2$ be $S(1, \mathcal{O}\!\mathcal{l})$ and $S(\phi_B - \phi_{B'}, \mathcal{O}\!\mathcal{l})$, respectively. Then the ranges of $(I + U)/2$ and $(I - U)/2$ are $S(f, \mathcal{O}\!\mathcal{l})$ and $S(f(\phi_B - \phi_{B'}), \mathcal{O}\!\mathcal{l})$, respectively. That they are independent is clear, from the independence of the induced pair, in Y_f. We have shown, then, that the contractive projections induced by reduced reflections form an independent pair. We consider, now, what happens when we begin with an independent pair of contractive projections.

Let $\{P, I - P\}$ be an independent pair, with ranges $S(f, \mathcal{O}\!\mathcal{l})$, $S(g, \mathcal{O}\!\mathcal{l})$, respectively. We consider, first, the case where $f = 1$; i.e., $P = E_{\mathcal{O}\!\mathcal{l}}$. Because of the independence of $\mathscr{B}(g)$ and $\mathcal{O}\!\mathcal{l}$, we may write, for $B \in \mathscr{B}(g)$,

$$\phi_B = m(B) + g\left(\int_B (|g|^p/g)\, dm\right).$$

It follows that g is constant off of B, and so g is two-valued. Also, $\mathscr{B}(g) = \{\varnothing, B, B', X\} \equiv \mathscr{B}$. We may then assume that $g\phi_B = \phi_B$. From the expression

$$\phi_B = m(B) + g(m(B)),$$

we see that $g\phi_{B'} = -\phi_{B'}$. Therefore $g = \phi_B - \phi_{B'}$. Since $\int g\, dm = 0$, it follows that $m(B) = m(B')$. We need the following result to complete this line of argument:

LEMMA 7. *Every set, E, in Σ, has the form $E = (A \cap B) \cup (C \cap B')$, for some A and C in $\mathcal{O}\!\mathcal{l}$, where B is the set described above.*

Proof. It suffices to show that, for every E in Σ, $E \cap B = A \cap B$ for some $A \in \mathcal{O}\!\mathcal{l}$. Suppose there is E for which this is false. Let $E^* = E \cap B$. Since $S(1, \mathcal{O}\!\mathcal{l}) \oplus S(g, \mathcal{O}\!\mathcal{l}) = L_p$, we have

$$\phi_{E^*} = E_{\mathcal{O}\!\mathcal{l}}(\phi_{E^*}) + gE_{\mathcal{O}\!\mathcal{l}}'(\phi_{E^*}/g).$$

Let $A = N(E_{\mathcal{O}\!\mathcal{l}}(\phi_{E^*}))$, and $J = A \cap (B - E)$. Then clearly $A \in \mathcal{O}\!\mathcal{l}$ and also $J \neq \varnothing$ (if $J = \varnothing$, then $E \cap B = A \cap B$, contrary to our assumption about E). Therefore

$$0 = \phi_J \phi_{E^*} = \phi_J E_{\mathcal{O}\!\mathcal{l}}(\phi_{E^*}) + \phi_J g E_{\mathcal{O}\!\mathcal{l}}'(\phi_{E^*}/g).$$

Both $E_{\mathcal{O}\!\mathcal{l}}(\phi_{E^*})$ and $E_{\mathcal{O}\!\mathcal{l}}'(\phi_{E^*}/g)$ are positive, with supports strictly larger than E^*. The above equation forces g to assume negative values on J, a contradiction, since $J \subseteq B$. Now, we define a set mapping, T, on Σ, by

$$T(E) = T((A \cap B) \cup (C \cap B')) = (A \cap B') \cup (C \cap B)$$

Then T is measure-preserving [since $E_{\mathcal{O}\!\mathcal{l}}(g) = 0$, $m(A \cap B) = m(A \cap B')$ for all $A \in \mathcal{O}\!\mathcal{l}$]. We obtain a reduced reflection, U, by defining

$$U(\phi_E) = \phi_{T(E)} \qquad \text{for all } E \in \Sigma,$$

and we see that $E_{\mathfrak{A}} = (I + U)/2$. We consider, now, the case where $f \neq 1$; i.e., $P \neq E_{\mathfrak{A}}$. On the space Y_f we define an operator, P', by

$$P'(k) = P(fk)/f.$$

It is easily seen that $\{P', I - P'\}$ is an independent pair of contractive projections, and that P' is a conditional expectation. The above argument, applied to P', tells us that $P' = (I + V)/2$ for some reduced reflection, V, on Y_f. On L_p, define U by

$$U(q) = fV(q/f).$$

It is clear that U is a reduced reflection on L_p, and, moreover, $P = (I + U)/2$. We have shown, then, that every independent pair of contractive projections is induced by a reduced reflection.

We assume, finally, that the third statement of the theorem is valid; there is a sub-σ-algebra, $\mathfrak{A}$, of Σ, and a set B, in Σ, such that, for every E in Σ,

$$E = (A \cap B) \cup (C \cap B'),$$

for some A, C in $\mathfrak{A}$. We shall show that there is a reduced reflection, U, on L_p, such that the invariant sub-space of U is generated by $\mathfrak{A}$. Define a set mapping, T, on Σ, by

$$T(E) = T((A \cap B) \cup (C \cap B')) = (A \cap B') \cup (C \cap B)$$

and let $|f|^p$ be the Radon–Nikodym derivative of the measure m^* [defined by $m^*(E) = m(T(E))$] with respect to m. Then, the mapping U, defined by

$$U(\phi_E) = f \cdot \phi_{T(E)}$$

extends to a reduced reflection on L_p, with invariant subspace, $S(1 + f, \mathfrak{A})$. We have proven Theorem 1.

We shall now prove the following:

THEOREM 2. *If $\{P, I - P\}$ is a total pair of contractive projections on L_p, then it is an independent pair.*

To prove this we need some notation and lemmas. Suppose the ranges are $S(f, \mathfrak{A})$ and $S(g, \mathfrak{A})$, respectively. Then, the projections P', and $I - P'$, on Y_f, defined by

$$P'(k) = P(fk)/f$$

form an independent pair iff $\{P, I - P\}$ is independent. Therefore, we shall consider only the case where $f = 1$; i.e., $P = E_{\mathcal{A}}$. Since P and $I - P$ are contractive projections on L_p, considered as a real Banach space, we may choose g to be real valued. Let $B = N(g^+)$, $B' = N(g^-)$ [where $g^+(x) = \max(g(x), 0)$]. The following lemma is needed:

LEMMA 8. *Every $E \in \Sigma$ has the form $E = (A \cap B) \cup (C \cap B')$ for some A, C in $\mathcal{A}$.*

Proof. The proof is identical with that of Lemma 7. In that proof we made no use of the independence of the pair, nor did we use any other information about the set, B, other than that g was positive on B.

Now, as in the case above, we define a set mapping, T, by

$$T(E) = T((A \cap B) \cup (C \cap B')) = (A \cap B') \cup (C \cap B).$$

It is easily shown that T is a regular set isomorphism on Σ. Define a measure, m^*, on Σ, by

$$m^*(E) = \tfrac{1}{2}m(A) + \tfrac{1}{2}m(C),$$

and let $|f|^p$ be the Radon–Nikodym derivative, dm^*/dm. With respect to the measure, $|f|^p m$, the mapping T is measure-preserving, and therefore induces a reflection, U', in Y_f, by

$$U'(\phi_E) = \phi_{T(E)} .$$

It is easy to see that U' is reduced, and that $U'(1) = 1$. The above theorem tells us that we can decompose Y_f as follows:

$$Y_f = S(1, \mathcal{A}) + S(\phi_B - \phi_{B'}, \mathcal{A}).$$

The projection onto $S(1, \mathcal{A})$ is $E_{\mathcal{A}}'$, and $\{E_{\mathcal{A}}', I - E_{\mathcal{A}}'\}$ is an independent pair in Y_f.

LEMMA 9. *If $h \in L_p$, then $E_{\mathcal{A}}'(h/f) = E_{\mathcal{A}}(h \cdot \bar{f}^{p-1})/E_{\mathcal{A}}(|f|^p)$.*

Proof. The right-hand term is $\mathcal{A}$-measurable. We show that it has the defining property of the conditional expectation, $E_{\mathcal{A}}'(h/f)$. Let $A \in \mathcal{A}$. Then

$$\int_A [E_{\mathcal{A}}(h\bar{f}^{p-1})/E_{\mathcal{A}}(|f|^p)]\, |f|^p\, dm$$

$$= \int_A E_{\mathcal{A}}(h\bar{f}^{p-1})\, dm, \qquad \text{since } E_{\mathcal{A}}(|f|^p) = 1 \text{ and } |f|^b\, dm = dm \text{ on } \mathcal{A},$$

$$= \int_A E_{\mathcal{A}}(h\, |f|^p/f)\, dm,$$

$$= \int_A (h/f)\, |f|^p\, dm.$$

LEMMA 10. *For $h \in L_p$,*

$$E_{\mathcal{A}}'(h/f) = E_{\mathcal{A}}(h\bar{f}^{p-1}).$$

Proof. This follows from Lemma 9, above, upon noticing that $E_{\mathcal{A}}(|f|^p) = 1$ (which is true, since, as above, the measure, m^*, is identical with m, on sets in $\mathcal{A}$).

LEMMA 11. *For all $h \in L_p$, $E_{\mathcal{A}}(h\bar{f}^{p-1})$ is in L_p.*

Proof. It is clearly in L_1. Since $E_{\mathcal{A}}(h\bar{f}^{p-1})$ is $\mathcal{A}$-measurable, we have

$$\begin{aligned}\int | E_{\mathcal{A}}(h\bar{f}^{p-1})|^p \, dm^* &= \int | E_{\mathcal{A}}(h\bar{f}^{p-1})|^p \, |f|^p \, dm \\ &= \int | E_{\mathcal{A}}'(h/f)|^p \, |f|^p \, dm \\ &= \int | h/f |^p \, |f|^p \, dm \\ &= \| h \|_p^p .\end{aligned}$$

LEMMA 12. *An $\mathcal{A}$-measurable function is in L_p iff it is in Y_f.*

Proof. This is a simple consequence of the equality of m and m^* on sets of $\mathcal{A}$.

By Lemma 12 the cycle $S(1, \mathcal{A}) \subseteq Y_f$ is in L_p, and since $m = m^*$ for sets in $\mathcal{A}$, we can see that $S(1, \mathcal{A})$ is closed in L_p. Similarly, any function in $S(\phi_B - \phi_{B'}, \mathcal{A}) \subseteq Y_f$ is shown to be in L_p, and this second cycle is also closed in L_p, for the same reason. Since $Y_k \cap L_p$ is dense in the latter, we conclude that L_p can be decomposed

$$L_p = S(1, \mathcal{A}) \oplus S(\phi_B - \phi_{B'}, \mathcal{A}).$$

It follows that the original g, in the representation of the range of $I - P$ could have been chosen to be $\phi_B - \phi_{B'}$. Since $\int_A g \, dm = 0$ for all $A \in \mathcal{A}$, we have $m(A \cap B) = m(A \cap B')$, and so T is measure-preserving with respect to m. It follows that $\mathcal{A}$ and $\mathcal{B}(g)$ are independent. We have proven Theorem 2.

From our discussion, above, concerning the decomposition of a reflection into a reduced reflection and the identity, restricted to complementary cycles of the type $S(\phi_E, \Sigma)$, $S(\phi_{E'}, \Sigma)$, we can conclude with the following

COROLLARY. *If P is a contractive projection on L_p, then $I - P$ is contractive iff there is a reflection, U, on L_p, such that $P = (I + U)/2$. The pair $\{P, I - P\}$ is total iff U is reduced.*

3. Generated Reflections

In this section we show that an arbitrary linear isometry on L_p gives rise, in a natural way, to a reflection.

If V is an isometry of L_p onto L_p, we let P_V be the contractive projection onto the invariant subspace of V. Since L_p is reflexive, the mean ergodic theorem implies that P_V is the limit, in the strong-operator topology, of the Cesàro sums, $(1/n+1)\sum_{i=0}^{n} V^i$.

Lemma 13. *If V is an isometry of L_p, then*

(1) *$P_V + P_{-V}$ is the contractive projection onto the invariant subspace of V^2;*

(2) *If f is in the range of P_V and g is in the range of P_{-V}, then*

$$\|f+g\| = \|f-g\|;$$

(3) *The operator $P_V - P_{-V}$ leaves the invariant subspace of V^2 pointwise fixed, and is a reflection, when restricted to this subspace.*

Proof. (1) This follows from a consideration of the Cesàro limits and is not difficult. (2) This is shown by sequence

$$\|f+g\| = \|V(f+G)\| = \|f-g\|.$$

To prove (3), we let f be in the range of $P_V - P_{-V}$, so that $f = g \pm h$, where $V(g) = g$, $V(h) = -h$. Then $V^2(f) = f$. From (2) we see that $P_V - P_{-V}$ is an isometry on the invariant subspace of V^2, and since $P_V P_{-V} = P_{-V} P_V = 0$, we have

$$(P_V - P_{-V})^2 = P_V + P_{-V} = I,$$

on the invariant subspace of V^2.

Theorem 3. *If V is an isometry of L_p onto L_p, with $V(1) = 1$, and if $L_p' = S(1, \mathscr{B})$, for $\mathscr{B}$, the sub-σ-algebra generated by the sets $E \in \Sigma$ such that $EV^j = V^j E$ for some $j = 2^i$, $i = 0, 1, 2, \ldots$, then there is an operator, W, with range, L_p', such that $\|Wf\| = \|E_{\mathscr{B}} f\|$ for all $f \in L_p$, and $W^2 = E_{\mathscr{B}}$; i.e., W is a reflection, on L_p'.*

Proof. By considering the sequence of contractive projections, $P_i = P_{V^j}$, $j = 2^i$, $i > 1$, and using Lemma 13, we obtain a sequence of operators, U_i, with $U_i^2 = P_i$, $U_i[L_p] = P_i[L_p]$, and $\|U_i f\| = \|P_i f\|$ for all f. It is easily shown that

$$U_{i+j} U_i = U_i,$$
$$U_i P_i = U_i.$$

From the marginale convergence theorem, the sequence $\{P_i\}$ converges, in the strong operator topology to a contractive projection, P', with range L_p'. This implies that the sequence $\{U_i\}$ also converges, since, for each f,

$$\begin{aligned}\| U_{i+j}f - U_if \| &\leqslant \| P_i(U_{i+j}f - U_if)\| + \|(I - P_i)(U_{i+j}f - U_if)\| \\ &= \| U_i(U_{i+j}f - U_if)\| + \| U_{i+j}f - P_iU_{i+j}f \| \\ &= \| P_{i+j}U_{i+j}f - P_iU_{i+j}f \| \\ &= \| U_{i+j}(P_{i+j}f - P_if)\| \\ &\leqslant \| U_{i+j} \| \, \| P_{i+j}f - P_if \|.\end{aligned}$$

Since $\| U_{i+j} \| \leqslant 2$, this last term can be made arbitrarily small. Let $W = \lim(i)U_i$. Then, for all f,

$$\begin{aligned}\| Wf \| &= \lim(i) \| U_if \| \\ &= \lim(i) \| P_if \| \\ &= \| P'f \|,\end{aligned}$$

and given $\epsilon > 0$, and i, j, k large enough,

$$\begin{aligned}\| W^2f - P'f \| - \epsilon &< \| U_iU_jf - P_kf \| \\ &= \| U_iU_jf - P_jf + P_jf - P_kf \| \\ &< \| U_iU_jf - P_jf \| + \| P_jf - P_kf \| \\ &= \| U_iU_jf - U_jU_jf \| + \| P_jf - P_kf \| \\ &< \| U_j \| \, \| U_if - U_jf \| + \| P_jf - P_kf \|.\end{aligned}$$

Since the U_i are increasing, it is clear that W has the required range.

References

[1] Ando, T. (1966). Contractive projections in L_p space. *Pacific J. Math.* **17** 391–405.
[2] Byrne, C., Cohen, H., and Sullivan, F. E. (1970). A note on contractive projections in L_p spaces, to appear.
[3] Cohen, H. and Sullivan, F. E. (1970). Projecting onto cycles in smooth, reflexive Banach spaces. *Pacific J. Math.* **34** 355–364.
[4] Köethe, G. (1969). *Topological Vector Spaces.* Springer-Verlag, New York.
[5] Lamperti, J. (1958). On the isometries of certain function spaces. *Pacific J. Math.* **8** 459–466.
[6] Lòeve, M. (1963). *Probability Theory.* Van Nostrand, New York.
[7] Sullivan, F. E. (1970). Structure of real L_p spaces. *J. Math. Anal. Appl.* **32** 421–429.