The Trans-Atlantic Cable

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Abstract

In 1815, at the end of the war with England, the US was a developing country, with most people living on small farms, eating whatever they could grow themselves. Only those living near navigable water could market their crops. Poor transportation and communication kept them isolated. By 1848, at the end of the next war, this time with Mexico, things were different. The US was a transcontinental power, integrated by railroads, telegraph, steamboats, the Erie Canal, and innovations in mass production and agriculture. In 1828, the newly elected President, Andrew Jackson, arrived in Washington by horse-drawn carriage; he left in 1837 by train. The most revolutionary change was in communication, where the recent advances in understanding electromagnetism produced the telegraph. It wasn't long before efforts began to lay a telegraph cable under the Atlantic Ocean, even though some wondered what England and the US could possibly have to say to one another.

The laying of the trans-Atlantic cable was, in many ways, the 19th century equivalent of landing a man on the moon, involving, as it did, considerable expense, too frequent failure, and a level of precision in engineering design and manufacturing never before attempted. From a scientific perspective, it was probably more difficult, given that the study of electromagnetism was in its infancy at the time.

In this note we discuss the efforts to develop an accurate mathematical model for transmission of signals through a long underwater cable, the leading role this model played in inspiring the technological innovation that eventually led to the success of the enormous effort, and the partial differential equation on which so much money depended.

1 Introduction

In 1815, at the end of the war with England, the US was a developing country, with most people living on small farms, eating whatever they could grow themselves. Only those living near navigable water could market their crops. Poor transportation and communication kept them isolated. By 1848, at the end of the next war, this time with Mexico, things were different. The US was a transcontinental power, integrated by railroads, telegraph, steamboats, the Erie Canal, and innovations in mass production and agriculture. In 1828, the newly elected President, Andrew Jackson, arrived in Washington by horse-drawn carriage; he left in 1837 by train. The most revolutionary change was in communication, where the recent advances in understanding electromagnetism produced the telegraph. It wasn't long before efforts began to lay a telegraph cable under the Atlantic Ocean, even though some wondered what England and the US could possibly have to say to one another.

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Early on, Faraday and others worried that sending a message across a vast distance would take a long time, but they reasoned, incorrectly, that this would be similar to filling a very long hose with water. What they did not realize initially was that, as William Thomson was to discover, the transmission of a pulse through an undersea cable was described more by a heat equation than a wave equation. This meant that a signal that started out as a sharp pulse would be spread out as time went on, making communication extremely slow. The problem was the increased capacitance with the ground.

Somewhat later, Oliver Heaviside realized that, when all four of the basic elements of the electrical circuit, the inductance, the resistance, the conductance to the ground and the capacitance to the ground, were considered together, it might be possible to adjust these parameters, in particular, to increase the inductance, so as to produce undistorted signals. Heaviside died in poverty, but his ideas eventually were adopted.

In 1859 Queen Victoria sent President Buchanan a 99 word greeting using an early version of the cable, but the message took over sixteen hours to be received. By 1866 one could transmit eight words a minute along a cable that stretched from Ireland to Newfoundland, at a cost of about 1500 dollars per word in today's money.

With improvements in insulation, using gutta percha, a gum from a tropical tree also used to make golf balls, and the development of magnetic alloys that increased the inductance of the cable, messages could be sent faster and more cheaply.

In this note we survey the development of the mathematics of the problem. We focus, in particular, on the partial differential equations that were used to describe the transmission problem. What we give here is a brief glimpse; more detailed discussion of this problem is found in the books by Körner [2], Gonzalez-Velasco [1], and Wylie [3].

2 The Electrical Circuit ODE

We begin with the ordinary differential equation that describes the horizontal motion of a block of wood attached to a spring. We let $x(t)$ be the position of the block relative to the equilibrium position $x = 0$, with $x(0)$ and $x'(0)$ denoting the initial position and velocity of the block. When an external force $f(t)$ is imposed, a portion of this force is devoted to overcoming the inertia of the block, a portion to compressing or stretching the spring, and the remaining portion to resisting friction. Therefore, the differential equation describing the motion is

$$mx''(t) + ax'(t) + kx(t) = f(t), \tag{2.1}$$

where m is the mass of the block, a the coefficient of friction, and k the spring constant.

The charge $Q(t)$ deposited on a capacitor in an electrical circuit due to an imposed electromotive force $E(t)$ is similarly described by the ordinary differential equation

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t). \tag{2.2}$$

The first term, containing the inductance coefficient L, describes the portion of the force $E(t)$ devoted to overcoming the effect of a change in the current $I(t) = Q'(t)$; here L is analogous to the mass m. The second term, containing the resistance coefficient R, describes that portion of the force $E(t)$ needed to overcome resistance to the current $I(t)$; now R is analogous to the friction coefficient a. Finally, the third term, containing the reciprocal of the capacitance C, describes the portion of $E(t)$ used to store charge on the capacitor; now $\frac{1}{C}$ is analogous to k, the spring constant.

3 The Telegraph Equation

The objective here is to describe the behavior of $u(x,t)$, the voltage at location x along the cable, at time t. In the beginning, it was believed that the partial differential equation describing the voltage would be the wave equation

$$u_{xx} = \alpha^2 u_{tt}.$$

If this were the case, an initial pulse

$$E(t) = H(t) - H(t-T)$$

would move along the cable undistorted; here $H(t)$ is the Heaviside function that is zero for $t < 0$ and one for $t \geq 0$. Thomson (later Sir William Thomson, and even later, Lord Kelvin) thought otherwise.

Thomson argued that there would be a voltage drop over an interval $[x, x+\Delta x]$ due to resistance to the current $i(x,t)$ passing through the cable, so that

$$u(x+\Delta x, t) - u(x,t) = -Ri(x,t)\Delta x,$$

and so

$$\frac{\partial u}{\partial x} = -Ri.$$

He also argued that there would be capacitance to the ground, made more significant under water. Since the apparent change in current due to the changing voltage across the capacitor is

$$i(x+\Delta x, t) - i(x,t) = -Cu_t(x,t)\Delta x,$$

we have

$$\frac{\partial i}{\partial x} = -C\frac{\partial u}{\partial t}.$$

Eliminating the $i(x,t)$, we can write

$$u_{xx}(x,t) = CRu_t(x,t), \tag{3.1}$$

which is the heat equation, not the wave equation.

4 Consequences of Thomson's Model

To see what Thomson's model predicts, we consider the following problem. Suppose we have a semi-infinite cable, that the voltage is $u(x,t)$ for $x \geq 0$, and $t \geq 0$, and that

$u(0,t) = E(t)$. Let $U(x,s)$ be the Laplace transform of $u(x,t)$, viewed as a function of t. Then, from Thomson's model we have

$$U(x,s) = \mathcal{L}(E)(s)e^{-\sqrt{CRs}x},$$

where $\mathcal{L}(E)(s)$ denotes the Laplace transform of $E(t)$. Since $U(x,s)$ is the product of two functions of s, the convolution theorem applies. But first, it is helpful to find out which function has for its Laplace transform the function $e^{-\alpha x\sqrt{s}}$. The answer comes from the following fact: the function

$$be^{-b^2/4t}/2\sqrt{\pi}t^{3/2}$$

has for its Laplace transform the function $e^{-b\sqrt{s}}$. Therefore, we can write

$$u(x,t) = \frac{\sqrt{CR}x}{2\sqrt{\pi}}\int_0^t E(t-\tau)\frac{e^{-CRx^2/4\tau}}{\tau\sqrt{\tau}}d\tau.$$

Now we consider two special cases.

4.1 Special Case 1: $E(t) = H(t)$

Suppose now that $E(t) = H(t)$, the Heaviside function. Using the substitution

$$z = CRx^2/4\tau,$$

we find that

$$u(x,t) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{CR}x/2\sqrt{\pi}} e^{-z^2}dz. \tag{4.1}$$

The function

$$\mathrm{erf}(r) = \frac{2}{\sqrt{\pi}}\int_0^r e^{-z^2}dz$$

is the well known *error function*, so we can write

$$u(x,t) = 1 - \mathrm{erf}\big(\frac{\sqrt{CR}x}{2\sqrt{t}}\big). \tag{4.2}$$

4.2 Special Case 2: $E(t) = H(t) - H(t-T)$

Now suppose that $E(t)$ is the pulse $H(t) - H(t-T)$. Using the results from the previous subsection, we find that, for $t > T$,

$$u(x,t) = \mathrm{erf}\big(\frac{\sqrt{CR}x}{2\sqrt{t-T}}\big) - \mathrm{erf}\big(\frac{\sqrt{CR}x}{2\sqrt{t}}\big). \tag{4.3}$$

For fixed x, $u(x,t)$ is proportional to the area under the function e^{-z^2}, over an interval that, as time goes on, moves steadily to the left and decreases in length. For small t the interval involves only large z, where the function e^{-z^2} is nearly zero and the integral is nearly zero. As t increases, the interval of integration moves to the left, so that the integrand grows larger, but the length of the interval grows smaller. The net effect is that the voltage at x increases gradually over time, and then decreases gradually; the sharp initial pulse is smoothed out in time.

5 Heaviside to the Rescue

It seemed that Thomson had solved the mathematical problem and discovered why the behavior was not wave-like. Since it is not really possible to reduce the resistance along the cable, and capacitance to the ground would probably remain a serious issue, particularly under water, it appeared that little could be done to improve the situation. But Heaviside had a solution.

Heaviside argued that Thomson had ignored two other circuit components, the leakage of current to the ground, and the self-inductance of the cable. He revised Thomson's equations, obtaining

$$u_x = -Li_t - Ri,$$

and

$$i_x = -Cu_t - Gu,$$

where L is the inductance and G is the coefficient of leakage of current to the ground. The partial differential equation governing $u(x,t)$ now becomes

$$u_{xx} = LCu_{tt} + (LG + RC)u_t + RGu, \tag{5.1}$$

which is the formulation used by Kirchhoff. As Körner remarks, never before had so much money been riding on the solution of one partial differential equation.

5.1 A Special Case: $G = 0$

If we take $G = 0$, thereby assuming that no current passes into the ground, the partial differential equation becomes

$$u_{xx} = LCu_{tt} + RCu_t, \tag{5.2}$$

or

$$\frac{1}{CL}u_{xx} = u_{tt} + \frac{R}{L}u_t. \tag{5.3}$$

If R/L could be made small, we would have a wave equation again, but with a propagation speed of $1/\sqrt{CL}$. This suggested to Heaviside that one way to obtain undistorted signaling would be to increase L, since we cannot realistically hope to change R. He argued for years for the use of cables with higher inductance, which eventually became the practice, helped along by the invention of new materials, such as magnetic alloys, that could be incorporated into the cables.

5.2 Another Special Case

Assume now that $E(t)$ is the pulse. Applying the Laplace transform method described earlier to Equation (5.1), we obtain

$$U_{xx}(x, s) = (Cs + G)(Ls + R)U(x, s) = \lambda^2 U(x, s),$$

from which we get

$$U(x, s) = A(s)e^{\lambda x} + \Big(\frac{1}{s}(1 - e^{-Ts}) - A(s)\Big)e^{-\lambda x}.$$

If it happens that $GL = CR$, we can solve easily for λ:

$$\lambda = \sqrt{CL}s + \sqrt{GR}.$$

Then we have

$$U(x, s) = e^{-\sqrt{GR}x}\frac{1}{s}(1 - e^{-Ts})e^{-\sqrt{CL}xs},$$

so that

$$u(x, t) = e^{-\sqrt{GR}x}\Big(H(t - x\sqrt{CL}) - H(t - T - x\sqrt{CL})\Big). \tag{5.4}$$

This tells us that we have an undistorted pulse that arrives at the point x at the time $t = x\sqrt{CL}$.

In order to have $GL = CR$, we need $L = CR/G$. Since C and R are more or less fixed, and G is typically reduced by insulation, L will need to be large. Again, this argues for increasing the inductance in the cable.

References

[1] Gonzalez-Velasco, E. (1996) *Fourier Analysis and Boundary Value Problems.* Academic Press.

[2] Körner, T. (1988) *Fourier Analysis.* Cambridge, UK: Cambridge University Press.

[3] Wylie, C.R. (1966) *Advanced Engineering Mathematics*. New York: McGraw-Hill.