Notes on Green's Theorem and Related Topics

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Abstract

Green's Theorem in two dimensions (Green-2D) has different interpretations that lead to different generalizations, such as Stokes's Theorem and the Divergence Theorem (Gauss's Theorem). Either of the latter two theorems can legitimately be called Green's Theorem for three dimensions.

Green's Theorem in two dimensions has several important corollaries. These involve normal derivatives and Laplacians and can be used to investigate properties of harmonic functions. Since the real and imaginary parts of analytic functions are harmonic, these identities are connected to the Cauchy Integral Theorems in complex function theory.

1 Introduction

Green's Theorem in two dimensions can be interpreted in two different ways, both leading to important generalizations, namely Stokes's Theorem and the Divergence Theorem. In addition, Green's Theorem has a number of corollaries that involve normal derivatives, Laplacians, and harmonic functions, and that anticipate results in analytic function theory, such as the Cauchy Integral Theorems. A good reference is the book by Flanigan [1].

1.1 Some Terminology

A subset D of R^2 is said to be *open* if, for every point x in D, there is $\epsilon > 0$, such that the ball centered at x, with radius ϵ is completely contained within D. The set D is *connected* if it is not the union of two disjoint non-empty open sets. The set D is said to be a *domain* if D is non-empty, open and connected. A subset B of R^2 is *bounded* if it is a subset of a ball of finite radius. The *boundary* of a set D, denoted

∂D, is the set of all points x, in D or not, such that every ball centered at x contains points in D and points not in D.

Because we shall be interested in theorems that relate the behavior of functions inside a domain to their behavior on the boundary of that domain, we need to limit our discussion to those domains that have nice boundaries. A *Jordan curve* is a piece-wise smooth closed curve that does not cross itself. A *Jordan domain* is a bounded domain, whose boundary consists of finitely many, say k, disjoint Jordan curves, parameterized in such a way that as a point moves around the curve with increasing parameter, the domain always lies to the left; this is *positive orientation.* Then the domain is called a *k-connected Jordan domain.* For example, a ball in R^2 is 1-connected, while an annulus is 2-connected; Jordan domains can have holes in them.

1.2 Arc-Length Parametrization

Let C be a curve in space with parameterized form

$$\mathbf{R}(t) = (x(t), y(t), z(t)).$$

For each t, let $s(t)$ be the distance along the curve from the point $\mathbf{R}(0)$ to the point $\mathbf{R}(t)$. The function $s(t)$ is invertible, so that we can also express t as a function of s, $t = t(s)$. Then $s(t)$ is called the *arc-length.* We can then rewrite the parametrization, using as the parameter the variable s instead of t; that is, the curve C can be described as

$$\mathbf{R}(s) = \mathbf{R}(t(s)) = (x(t(s)), y(t(s)), z(t(s))). \tag{1.1}$$

Then

$$\mathbf{R}'(t) = \frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{ds}\frac{ds}{dt} = (\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds})\frac{ds}{dt}. \tag{1.2}$$

The vector

$$\mathbf{T}(s) = \frac{d\mathbf{R}}{ds} = (\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}) \tag{1.3}$$

has length one, since

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{1.4}$$

and $v = \frac{ds}{dt}$, the *speed* along the curve, satisfies

$$(\frac{ds}{dt})^2 = (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2. \tag{1.5}$$

2 Green's Theorem in Two Dimensions

Green's Theorem for two dimensions relates double integrals over domains D to line integrals around their boundaries ∂D. Theorems such as this can be thought of as two-dimensional extensions of integration by parts. Green published this theorem in 1828, but it was known earlier to Lagrange and Gauss.

Theorem 2.1 (Green-2D) *Let $P(x, y)$ and $Q(x, y)$ have continuous first partial derivatives for (x, y) in a domain Ω containing both Jordan domain D and ∂D. Then*

$$\oint_{\partial D} P dx + Q dy = \int\int_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dx dy. \tag{2.1}$$

Let the boundary ∂D be the positively oriented parameterized curve

$$\mathbf{R}(t) = (x(t), y(t)).$$

Then, for each t, the vector

$$\mathbf{R}'(t) = (x'(t), y'(t))$$

is tangent to the curve at the point $\mathbf{R}(t)$. The vector

$$\mathbf{N}(t) = (y'(t), -x'(t))$$

is perpendicular to $\mathbf{R}'(t)$ and is outwardly normal to the curve at the point $\mathbf{R}(t)$. The integrand on the left side of Equation (2.1) can be written in two ways:

$$P dx + Q dy = (P, Q) \cdot \mathbf{R}'(t) dt, \tag{2.2}$$

or as

$$P dx + Q dy = (Q, -P) \cdot \mathbf{N}(t) dt. \tag{2.3}$$

In Equation (2.2) we use the dot product of the vector field $\mathbf{F} = (P, Q)$ with a tangent vector; this point of view will be extended to Stokes's Theorem. In Equation (2.3) we use the dot product of the vector field $\mathbf{G} = (Q, -P)$ with a normal vector; this formulation of Green's Theorem, also called Gauss's Theorem in the plane, will be extended to the Divergence Theorem, also called Gauss's Theorem in three dimensions. Either of these extensions therefore can legitimately be called Green's Theorem in three dimensions.

3 Proof of Green-2D

First, we compute the line integral $\oint Pdx + Qdy$ around a small rectangle in D and then sum the result over all such small rectangles in D. For convenience, we assume the parameter s is arc-length.

Consider the rectangle with vertices (x_0, y_0), $(x_0+\Delta x, y_0)$, $(x_0+\Delta x, y_0+\Delta y)$, and $(x_0, y_0+\Delta y)$, where Δx and Δy are very small positive quantities. The boundary curve is counter-clockwise. The line integrals along the four sides are as follows:

- The right side:

$$\int_{y_0}^{y_0+\Delta y} Q(x_0+\Delta x, y)dy; \tag{3.1}$$

- The top:

$$\int_{x_0+\Delta x}^{x_0} P(x, y_0+\Delta y)dx; \tag{3.2}$$

- The left side:

$$\int_{y_0+\Delta y}^{y_0} Q(x_0, y)dy; \tag{3.3}$$

- The bottom:

$$\int_{x_0}^{x_0+\Delta x} P(x, y_0)dx. \tag{3.4}$$

Now consider the double integral

$$\int\int_{\Delta}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx\,dy, \tag{3.5}$$

where Δ denotes the infinitesmal rectangular region. We write the first half of this integral as

$$\begin{aligned}
&\int_{y_0}^{y_0+\Delta y}\Big(\int_{x_0}^{x_0+\Delta x} Q_x(x,y)dx\Big)dy,\\
&= \int_{y_0}^{y_0+\Delta y}\Big(Q(x_0+\Delta x, y) - Q(x_0, y)\Big)dy,\\
&= \int_{y_0}^{y_0+\Delta y} Q(x_0+\Delta x, y)dy - \int_{y_0}^{y_0+\Delta y} Q(x_0, y)dy,
\end{aligned}$$

which is the sum of the two integrals in lines 3.1 and 3.3. In the same way, we can show that the second half of the double integral is equal to the line integrals along the top and bottom of Δ.

Now consider the contributions to the double integral

$$\int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx\, dy, \tag{3.6}$$

which is the sum of each of the double integrals over all the small rectangles Δ in D. When we add up the contributions of all these infinitesimal rectangles, we need to note that rectangles adjacent to one another contribute nothing to the line integral from their shared edge, since the unit outward normals are opposite in direction. Consequently, the sum of all the line integrals around the small rectangles reduces to the line integral around the boundary of D, since this is the only curve without any shared edges. The double integral in Equation (3.6) is then the line integral around the boundary only, which is the assertion of Green-2D.

4 Extension to Three Dimensions

4.1 Stokes's Theorem

The first extension of Green-2D to three dimensions that we shall discuss is Stokes's Theorem. The statement of Stokes's Theorem involves a curve C in space and a surface S that is a *capping surface* for C. A good illustration of a capping surface is the soap bubble formed when we blow air through a soapy ring; the ring is C and the bubble formed is S.

Theorem 4.1 *Let C be a Jordan curve in space with unit tangent $\mathbf{T}(s) = \frac{d\mathbf{R}}{ds}$ and S a capping surface for C, with outward unit normal vectors $\mathbf{n}(s)$. Let $\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ be a vector field. Then*

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \int\int_S \mathbf{n} \cdot \mathrm{curl}(\mathbf{F}) dS. \tag{4.1}$$

The curl of $\mathbf{F}$ is the vector field

$$\mathrm{curl}(\mathbf{F}) = (R_y - Q_z, P_z - R_x, Q_x - P_y), \tag{4.2}$$

where $R_y = \frac{\partial R}{\partial y}$.

Suppose that $\mathbf{F} = (P, Q, 0)$ and the surface S is a Jordan domain D in R^2, with $C = \partial D$. Then

$$\mathrm{curl}(\mathbf{F}) = (0, 0, Q_x - P_y),$$

and $\mathbf{n} = (0, 0, 1)$. Therefore,

$$\mathbf{n} \cdot \mathrm{curl}(\mathbf{F}) = Q_x - P_y.$$

Also,

$$\mathbf{F} \cdot \mathbf{T} = Pdx + Qdy.$$

We see then that Stokes's Theorem has Green-2D as a special case.

4.2 The Divergence Theorem

Equation (2.3) suggests that we consider surface integrals of functions having the form $\mathbf{F} \cdot \mathbf{n}$, where $\mathbf{n}$ is the outward unit normal to the surface at each point. The Divergence Theorem, also called Gauss's Theorem in three dimensions, is one result in this direction.

Theorem 4.2 *Let S be a closed surface enclosing the volume V. Let $\mathbf{F} = (P, Q, R)$ be a vector field with divergence*

$$\text{div}(\mathbf{F}) = P_x + Q_y + R_z.$$

Then

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int\int\int_V \text{div}(\mathbf{F})\, dV. \tag{4.3}$$

If we let $R = 0$ and imagine the volume shrinking down to a two-dimensional planar domain D, with S compressing down to its boundary, ∂D, the unit normal vector becomes

$$\mathbf{n} = (\frac{dy}{ds}, -\frac{dx}{ds}),$$

and Equation (4.3) reduces to Equation (2.1).

5 Corollaries of Green-2D

5.1 Green's First Identity

Let $u(x, y)$ be a differentiable real-valued function of two variables, with gradient

$$\nabla u(x, y) = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}).$$

Let D be a Jordan domain with boundary $C = \partial D$. The directional derivative of u, in the direction of the unit outward normal $\mathbf{n}$, is

$$\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}.$$

When the curve C is parameterized by arc-length, the unit outward normal is

$$\mathbf{n} = (\frac{dy}{ds}, -\frac{dx}{ds}),$$

and

$$\oint_C \frac{\partial u}{\partial \mathbf{n}}\, ds = \oint_C -u_y dx + u_x dy. \tag{5.1}$$

Theorem 5.1 (Green I) *Let $\nabla^2 q$ denote the Laplacian of the function $q(x, y)$, that is,*

$$\nabla^2 q = q_{xx} + q_{yy}.$$

Then

$$\int\int_D (\nabla p) \cdot (\nabla q)\, dxdy = \oint_C p \frac{\partial q}{\partial \mathbf{n}}\, ds - \int\int_D p \nabla^2 q\, dxdy. \tag{5.2}$$

Proof: Evaluate the line integral using Green-2D, with $P = -pq_y$ and $Q = pq_x$. ∎

5.2 Green's Second Identity

An immediate corollary is Green's Second Identity (Green II).

Theorem 5.2 (Green II)

$$\oint_C p \frac{\partial q}{\partial \mathbf{n}} - q \frac{\partial p}{\partial \mathbf{n}}\, ds = \int\int_D p \nabla^2 q - q \nabla^2 p\, dxdy. \tag{5.3}$$

5.3 Inside-Outside Theorem

The Inside-Outside Theorem, which is a special case of Gauss's Theorem in the plane, follows immediately from Green II.

Theorem 5.3 (Inside-Outside Theorem)

$$\oint_C \frac{\partial q}{\partial \mathbf{n}}\, ds = \int\int_D \nabla^2 q\, dxdy. \tag{5.4}$$

5.4 Green's Third Identity

Green's Third Identity (Green III) is more complicated than the previous ones. Let w be any point inside the Jordan domain D in R^2 and hold w fixed. For variable z in the plane, let $r = |z - w|$. A function is said to be *harmonic* if its Laplacian is identically zero. Using the Inside-Outside Theorem, we can show that the function

$p(z) = \log r$ is harmonic for any z in any domain that does not contain w. With $z = (x, y)$ and $w = (a, b)$, we have

$$r^2 = (x - a)^2 + (y - b)^2,$$

so that

$$p(z) = p(x, y) = \frac{1}{2} \log\left((x - a)^2 + (y - b)^2\right).$$

Then

$$p_x(x, y) = \frac{x - a}{(x - a)^2 + (y - b)^2},$$

$$p_{xx}(x, y) = \frac{(y - b)^2 - (x - a)^2}{((x - a)^2 + (y - b)^2)^2},$$

$$p_y(x, y) = \frac{y - b}{(x - a)^2 + (y - b)^2},$$

and

$$p_{yy}(x, y) = \frac{(x - a)^2 - (y - b)^2}{((x - a)^2 + (y - b)^2)^2}.$$

Clearly, we have

$$p_{xx} + p_{yy} = 0,$$

and so p is harmonic in any region not including w.

The theorem is the following:

Theorem 5.4 (Green III)

$$q(w) = \frac{1}{2\pi} \int\int_D \log r \, \nabla^2 q \, dxdy - \frac{1}{2\pi} \oint_C \log r \, \frac{\partial q}{\partial \mathbf{n}} \, ds + \frac{1}{2\pi} \oint_C q \frac{\partial \log r}{\partial \mathbf{n}} \, ds. \qquad (5.5)$$

The two line integrals in Equation (5.5) are known as the *logarithmic single-layer potential* and *logarithmic double-layer potential*, respectively, of the function q.

Notice that we cannot apply Green II directly to the domain D, since $\log r$ is not defined at $z = w$. The idea is to draw a small circle C' centered at w, with interior D' and consider the new domain that is the original D, without the ball D' around w; the new domain has a hole in it, but that is acceptable. Then apply Green II, and finally, let the radius of the ball go to zero. The key step in the calculation is to show that

$$\frac{\partial p}{\partial \mathbf{n}} = \nabla p \cdot \mathbf{n} = \frac{1}{\rho},$$

where ρ is the radius of the small circle C' centered at w. Then

$$\oint_{C'} q \frac{\partial p}{\partial \mathbf{n}} ds = \frac{1}{\rho} \oint_{C'} q ds,$$

which, as the radius of C' goes to zero, is just $2\pi q(w)$.

Equation (5.5) tells us that if q is a harmonic function in D, then its value at any point w inside D is completely determined by what the functions q and $\frac{\partial q}{\partial \mathbf{n}}$ do on the boundary C. Note, however, that the normal derivative of q depends on values of q near the boundary, not just on the boundary. In fact, $q(w)$ is completely determined by q alone on the boundary, via

$$q(w) = -\frac{1}{2\pi}\oint_C q(z)\frac{\partial}{\partial \mathbf{n}}G(z,w)\,ds,$$

where $G(z,w)$ is the *Green's function* for the domain D.

6 Application to Complex Function Theory

If $f(z)$ is an analytic function of the complex variable z, then the real and imaginary parts of $f(z)$ are harmonic functions. Using Green III, we can obtain *Cauchy's Integral Formula*, which shows that the value of f at any point w within the domain D is determined by the value of the function at the points z of the boundary:

$$f(w) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-w}dz. \tag{6.6}$$

This formula occurs in complex function theory under slightly weaker assumptions than we use here. We shall assume that $f(z)$ is continuously differentiable, so that the real and imaginary parts of f are continuously differentiable. In complex function theory, it is shown that the continuity of $f'(z)$ is a consequence of analyticity. According to complex function theory, we may, without loss of generality, consider only the case in which C is the circle of radius ρ centered at w, and D is the region enclosed by C, which is what we shall do.

The function $f(z)$ can be written in terms of its real and imaginary parts, as

$$f(z) = u(x,y) + iv(x,y), \tag{6.7}$$

for $z = x + iy$. When $f(z)$ is analytic, the functions u and v satisfy the Cauchy-Riemann Equations,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y},\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x},\end{aligned} \tag{6.8}$$

from which it follows easily that both u and v are harmonic.

We know from Green III that

$$u(w) = \frac{1}{2\pi}\int\int_D \log r\,\nabla^2 u\,dxdy - \frac{1}{2\pi}\oint_C \log r\,\frac{\partial u}{\partial \mathbf{n}}\,ds + \frac{1}{2\pi}\oint_C u\frac{\partial \log r}{\partial \mathbf{n}}\,ds, \tag{6.9}$$

with a similar expression involving v. Because u is harmonic, Equation (6.9) reduces to

$$u(w) = -\frac{1}{2\pi}\oint_C \log r\,\frac{\partial u}{\partial \mathbf{n}}\,ds + \frac{1}{2\pi}\oint_C u\frac{\partial \log r}{\partial \mathbf{n}}\,ds\,, \tag{6.10}$$

with a similar expression involving the function v.

Consider the first line integral in Equation (6.10),

$$\frac{1}{2\pi}\oint_C \log r\,\frac{\partial u}{\partial \mathbf{n}}\,ds. \tag{6.11}$$

Since $r = \rho$ for all z on C, this line integral becomes

$$\frac{1}{2\pi}\log\rho\oint_C \frac{\partial u}{\partial \mathbf{n}}\,ds. \tag{6.12}$$

But, by the Inside-Outside Theorem, and the fact that u is harmonic, we know that

$$\oint_C \frac{\partial u}{\partial \mathbf{n}}\,ds = \int\int_D \nabla^2 u\,dxdy\; = 0. \tag{6.13}$$

So we need only worry about the second line integral in Equation (6.10), which is

$$\frac{1}{2\pi}\oint_C u\frac{\partial \log r}{\partial \mathbf{n}}\,ds. \tag{6.14}$$

We need to look closely at the term

$$\frac{\partial \log r}{\partial \mathbf{n}}.$$

First, we have

$$\frac{\partial \log r}{\partial \mathbf{n}} = \frac{1}{2}\frac{\partial \log r^2}{\partial \mathbf{n}}. \tag{6.15}$$

The function $\log r^2$ can be viewed as

$$\log r^2 = \log\Big(\mathbf{a}\cdot\mathbf{a}\Big), \tag{6.16}$$

where $\mathbf{a}$ denotes $z - w$, thought of as a vector in R^2. Then

$$\nabla \log r^2 = \nabla \log\Big(\mathbf{a}\cdot\mathbf{a}\Big) = 2\frac{\mathbf{a}}{||\mathbf{a}||}. \tag{6.17}$$

Because C is a circle centered at w, the unit outward normal at z on C is

$$\mathbf{n} = \frac{\mathbf{a}}{||\mathbf{a}||}. \tag{6.18}$$

Putting all this together, we find that

$$\frac{\partial \log r}{\partial \mathbf{n}} = \frac{1}{||\mathbf{a}||} = \frac{1}{|z-w|}. \tag{6.19}$$

Therefore, Green III tells us that

$$u(w) = \frac{1}{2\pi} \oint_C \frac{u(z)}{|z-w|} ds, \tag{6.20}$$

with a similar expression involving v. There is one more step we must take to get to the Cauchy Integral Formula.

We can write $z - w = \rho e^{i\theta}$ for z on C. Therefore,

$$\frac{dz}{d\theta} = \rho i e^{i\theta}. \tag{6.21}$$

The arc-length s around the curve C is $s = \rho\theta$, so that

$$\frac{ds}{d\theta} = \rho. \tag{6.22}$$

Therefore, we have

$$\theta = \frac{s}{\rho}, \tag{6.23}$$

and

$$z - w = \rho e^{is/\rho}. \tag{6.24}$$

Then,

$$\frac{dz}{ds} = i e^{is/\rho}, \tag{6.25}$$

or

$$ds = \frac{1}{i} e^{-i\theta} dz. \tag{6.26}$$

Substituting for ds in Equation (6.20) and in the corresponding equation involving v, and using the fact that

$$|z-w| e^{i\theta} = z - w, \tag{6.27}$$

we obtain Cauchy's Integral Formula (6.6).

References

[1] Flanigan, F. (1983) *Complex Variables: Harmonic and Analytic Functions*, Dover Publ.