Richard Kaufman
Homework #7
Applied Math 1
Typed using MathType

OPERATIONS WITH MATRICES

15.49] Find x and y such that $\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} 8 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Book Answer:

$x=-33/5,\ y=-26/5$

15.50] If A and B are square matrices such that $AB=0$, prove that we can have $A\neq 0,\ B\neq 0$. Is the result true for non-square matrices?

15.51] If $AB=AC$, is it true that $B=C$? Explain.

15.53] A *linear transformation* from an (x_1,x_2) to a (y_1,y_2) coordinate system is defined as $y_1=a_{11}x_1+a_{12}x_2,\ y_2=a_{21}x_1+a_{22}x_2$. ($a$) If $Y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ X=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ show that the transformation can be written $Y=AX$. (b) If $X=BU$ where $X=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ U=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ show that $x_1=b_{11}u_1+b_{12}u_2,\ x_2=b_{21}u_1+b_{22}u_2$. Thus obtain y_1,y_2 in terms of u_1,u_2 and explain how you can use the approach to motivate a definition of AB.

15.55] Let P [Fig. 15-1, below] have coordinates (x_1y) relative to an xy coordinate system and (x',y') relative to an $x'y'$ coordinate system which is rotated through an angle θ relative to the xy coordinate system. (a) Prove that the relationship between the coordinates or transformation from (x,y) to (x',y') is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(b) Show that the square matrix in (a) is skew-symmetric.

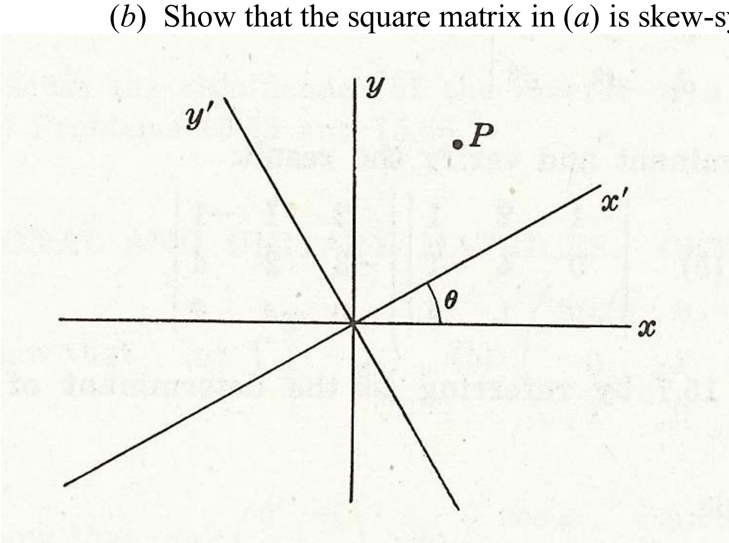

Fig. 15-1

15.58] Let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$. ($a$) Show that $X^T AX = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k$, which is called a *quadratic form* in $x_1,\ldots,x_n$. (b) Show that if A is a real symmetric matrix, i.e. $a_{jk} = a_{kj}$, then $X^T AX = a_{11}x_1^{\ 2} + a_{22}x_2^{\ 2} + \cdots + a_{nn}x_n^{\ 2} + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \cdots$, which is called a *symmetric quadratic form*. (c) What does the quadratic form become if A is skew-symmetric?

15.59] Write the quadratic forms (a) $4x_1^{\ 2} - 6x_1x_2 + 3x_2^{\ 2}$, ($b$) $x_1^{\ 2} - 2x_2^{\ 2} + 4x_3^{\ 2} + 2x_1x_2 - 6x_1x_3 + 4x_2x_3$ in terms of matrices.

15.60] If A is Hermitian or skew-Hermitian, the quadratic form $\bar{X}^T AX$ is called a Hermitian or skew-Hermitian form respectively. Prove that for every choice of X (a) the value of a Hermitian form is always real, (b) the value of a skew-Hermitian form is zero or pure imaginary.

15.61] Prove that every square matrix C can be written as $A+B$ where A is Hermitian and B is skew-Hermitian.

ORTHOGONAL AND UNITARY MATRICES. ORTHOGONAL VECTORS

15.87] Find a unit vector which is orthogonal to each of the vectors $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$.

SYSTEMS OF LINEAR EQUATIONS

15.93] Given the system of equations $\begin{cases} 3x_1 + 2x_2 - 4x_3 = 2 \\ 2x_1 - 3x_2 + x_3 = -1 \end{cases}$ show that any two of x_1, x_2, x_3 can be solved in terms of the remaining one and thus that there are infinitely many solutions.

EIGENVALUES AND EIGENVECTORS

15.96a] Find the eigenvalues and corresponding eigenvectors for .. the following matrix.

(a) $\begin{pmatrix} 2 & 2 \\ -1 & 5 \end{pmatrix}$

Book Answer:

(a) 3, 4, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

15.98] (a) Prove that if the eigenvalues of a matrix A are $\lambda_1, \lambda_2, \ldots$, then the eigenvalues of A^2 are $\lambda_1^{\ 2}, \lambda_2^{\ 2}, \ldots$. ($b$) Generalize the result in (a).

15.99] Prove that the eigenvalues of a skew-Hermitian matrix [or skew-symmetric real matrix] are either zero or pure imaginary.

15.107] (a) Find a transformation which removes the xy term in $x^2 + xy + y^2 = 16$ and (b) give a geometric interpretation of the result.

15.108] Discuss the relationship between Problem 15.107 and the problem of finding the maximum or minimum of $x^2 + y^2$ subject to the condition $x^2 + xy + y^2 = 16$. [*Hint*. Use the method of Lagrange multipliers.]