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ARCS DEFINED BY ONE-PARAMETER SEMIGROUPS OF OPERATORS IN BANACH SPACES WITH THE RADON-NIKODYM PROPERTY

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ABSTRACT. It is shown that a recent theorem of Junghenn and Taam concerning the domain of the infinitesimal generator of a strongly continuous one-parameter semigroup of operators on a reflexive, locally convex topological vector space remains valid if the domain of the operators is a Banach space with the Radon-Nikodym property. A partial result is obtained for general Banach spaces.

The following theorem is proved in a recent paper by Junghenn and Taam:

Theorem. *Let X be a reflexive, locally convex topological vector space. Let $T(t)$, for $t \geq 0$, be a strongly continuous semigroup of operators on X, such that for all $c > 0$, $T(c)$ is an isomorphism* (*into*), *and let A be the infinitesimal generator of the semigroup. The following are equivalent*:

(1) *x is in the domain of A*;

(2) *$T(\ )x$ is absolutely continuous on the interval $[0, c]$ for any $c > 0$*;

(3) *$T(\ )x$ is of bounded variation on the interval $[0, c]$ for any $c > 0$.*

In the absence of reflexivity, each condition implies the next. It is the purpose of this note to show that if X is a Banach space, then (2) and (3) are equivalent, and if, in addition, X has the Radon-Nikodym property, then all three are equivalent. Among spaces with the Radon-Nikodym property are reflexive Banach spaces and separable dual Banach spaces. For a discussion of the Radon-Nikodym property, see the paper of Rieffel [2]. Recent results characterizing Banach spaces with this property are to be found in [4], [5], and [6].

To prove the above assertions we need two lemmas. Our notation is that of [1]. Terms not defined in this note are as used in that paper.

Lemma 1. *Let X be a Banach space, and $x \in X$, for which* (3) *holds. If $c > 0$ and L is the total variation of $T(\ )x$ on $[0, c]$, then $\|T(c + h)x - T(c)x\|/h \leq KL/(c - h)$ for all $0 < h < c$, where K is a constant not dependent upon the choice of h* (*but dependent upon c*).

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Proof. Choose n so that $nh \leq c < (n+1)h$, and let $1 \leq j \leq n$. Pick K so that $\|T(t)x\| \leq K\|x\|$ for t in $[0, c]$. Then

$$\|T(c+h)x - T(c)x\| = \|T(c+h-jh)\{T(jh)x - T(jh-h)x\}\| \leq K\|T(jh)x - T(jh-h)x\|.$$

Therefore

$$n\|T(c+h)x - T(c)x\| \leq \sum_1^n K\|T(jh)x - T(jh-h)x\| \leq KL.$$

The assertion follows since we have $c - h < nh$.

Lemma 2. *Under the same assumptions as Lemma* 1, *the set* $\{\|T(h)x - x\|/h \mid 0 < h \leq c\}$ *is bounded.*

Proof. We have

$$\|T(h)x - x\|/h = \|T(c)^{-1}\{T(c+h)x - T(c)x\}\|/h \leq M\|T(c+h)x - T(c)x\|/h,$$

for $M = \|T(c)^{-1}\|$. Then $\|T(h)x - x\|/h \leq MKL/(c-h)$, and if $0 < h < c/2$ we have $\|T(h)x - x\|/h \leq 2MKL/c = Q$. Clearly, for h in $[c/2, c]$, the numbers are bounded.

We prove now that (3) implies (2) in any Banach space. Let $e > 0$ be given and let (a_i, b_i), $i = 1, 2, \ldots, n$, be any collection of nonoverlapping intervals in $[0, c]$. If we first assume that $b_i - a_i < c/2$, then

$$\|T(b_i)x - T(a_i)x\| = \|T(a_i)\{T(b_i - a_i)x - x\}\| \leq K\|T(b_i - a_i)x - x\|.$$

Therefore the choice of $d < e/QK$ gives $\Sigma_1^n \|T(b_i)x - T(a_i)x\| < e$ if $\Sigma_1^n (b_i - a_i) < d$.

Banach spaces X with the Radon-Nikodym property are precisely those spaces with the property that every function of bounded variation from the real line into X is differentiable almost everywhere. This is a classical result due to Bochner and Taylor [3]. If we assume that X has this property, then there is a $\bar{t} > 0$ at which point the function $T(\ )x$ is differentiable, if (3) holds. Then, as h approaches 0, $[T(\bar{t}+h)x - T(\bar{t})x]/h$ approaches a limit. It follows that $[T(h)x - x]/h$ does also, since it is equal to $[T(\bar{t})^{-1}\{\{T(\bar{t}+h)x - T(\bar{t})x\}/h\}]$. Therefore, if X is assumed to have the Radon-Nikodym property, the three conditions given in the Theorem are equivalent.

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