Barrow’s Proof of the Fundamental Theorem of Calculus

Barrow provides a geometric proof of the following theorem:

**Theorem 0.1** Suppose that $g(x)$ is continuous and strictly increasing for $0 \leq x < +\infty$ and $g(0) = 0$. Let $f(x) = \int_0^x g(t)dt$. Then $f'(x) = g(x)$.

Consider the diagram below. Above the x-axis we have the curve $OP$, which is the graph of the function $y = f(x)$ and below the x-axis the curve $OQ$, the graph of $y = -g(x)$. That is, the length of the segment $XP$ is the area between the x-axis and the graph from $O$ to $Q$. Pick point $T$ so that the slope of line $PT$ is equal to the length $XQ$; in other words, the slope of the line $PT$ equals $g(X)$. To prove the theorem we need to show that the line $PT$ is tangent to the upper graph at the point $P$.

Select $R$ on the curve $OP$ and $S$ on $PT$ with $RS$ parallel to the x-axis. We show that $R$ and $S$ are distinct points, so that the line $PT$ cannot intersect the curve $OP$ more than once.

Triangle $PSU$ is similar to $PTX$ so $PU/SU = PX/SX = XQ$. Therefore, $PU = SU \cdot XQ$.

Since $UX = VR$ is the area of sector $OVW$ and $PX$ is the area of sector $OXQ$ we have that $PU$, which is the area of region $VWQX$, is less than the area of region $VZQX$, which equals $RU \cdot XQ$. Therefore, $PU < RU \cdot XQ$, while $PU = SU \cdot XQ$. It follows that $RU > SU$ and $R$ and $S$ are distinct. This completes the proof.