Notes on Fourier Coefficients as Data

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1 Measuring the Fourier Coefficients

Suppose that $f(x)$ is a real or complex function defined for $|x| \leq L$, with Fourier series representation

$$f(x) = \sum_{m=-\infty}^{\infty} c_m \exp(i\frac{m\pi}{L}x). \tag{1.1}$$

Then the Fourier coefficients c_m are

$$c_m = \frac{1}{2L}\int_{-L}^{L} f(x)\exp(-i\frac{m\pi}{L}x)dx. \tag{1.2}$$

To obtain the Fourier coefficients we need to know $f(x)$ for all x in the interval $[-L, L]$. In a number of applications, in particular, in remote sensing, the function $f(x)$ is unknown and must be estimated from measured data. It is often the case that what we measure are some of the c_m, from which we must estimate $f(x)$.

2 A Remote-Sensing Example

Imagine that each point x in the interval $[-L, L]$ emits a signal at frequency ω, with signal strength $f(x)$ that depends on x. The signal could be reflected radio waves, as in radar, reflected or emitted acoustic waves, as in passive or active sonar, or just reflected light, from, say, the moon. In practice, such problems tend to be two- or three-dimensional, but for simplicity, we consider only the one-dimensional case here. The signal coming from the point x is the function of time t given by

$$s(t; x) = f(x)\exp(i\omega t), \tag{2.1}$$

where each $f(x)$ is a complex number and includes both magnitude $|f(x)|$ and phase $\phi(x)$, that is

$$f(x) = |f(x)| \exp(i\phi(x)).$$

We are interested in what is received at a point P that is a large distance away from the sources of the signals. The situation is pictured in Figure 1 (click on Figure 1 on website).

The line from the point $x = 0$ to the point P makes an angle of θ with the positive x-axis. The distance D from $x = 0$ to P is much larger than A, and so we say that P is in the *far-field*. The far-field assumption will permit us to make a useful approximation of the distance from each x to P.

The far-field assumption allows us to say that the distance from x to P is approximately $D - x\cos\theta$. With c the speed of propagation of the signal, the time required for the signal from x to reach P is $\frac{1}{c}(D - x\cos\theta)$. Therefore, at time t, P receives from x the value of the signal at the previous time $t - \frac{1}{c}(D - \cos\theta)$. Therefore, at time t, the point P receives from all the points in the interval $[-L, L]$

$$\exp(i\omega t)\exp(-i\frac{D\omega}{c})\int_{-L}^{L} f(x)\exp(i\frac{\omega\cos\theta}{c}x)dx.$$

The integral is the value of the Fourier transform of $f(x)$, at the point $\frac{\omega\cos\theta}{c}$.

To obtain the Fourier coefficient c_m we need to select the point P so that

$$\frac{\omega\cos\theta}{c} = \frac{-m\pi}{L},$$

or

$$\cos\theta = -m\frac{\pi c}{\omega L}.$$

Since $|\cos\theta| \leq 1$, it follows that

$$|m| \leq \frac{L\omega}{\pi c} = 2\frac{L}{\lambda},$$

where $\lambda = \frac{2\pi c}{\omega}$ is the wavelength. Consequently, we can measure only a finite number of the Fourier coefficients. This leads to the problem of estimating $f(x)$ from finitely many of the c_m.

3 Another Remote-Sensing Example

A waveform propagating at speed c in two dimensions is often modeled as a function $u(t, x, y)$ of time t and spatial variables x and y satisfying the two-dimensional wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}),$$

where the subscripts denote partial differentiation. One solution of this equation is the *single-frequency planewave* field.

The single-frequency planewave field is a function of the form

$$u(t,x,y) = \exp(i\omega t)\exp(i(\frac{\omega\cos\theta}{c}x + \frac{\omega\sin\theta}{c}y)),$$

where ω is some fixed frequency, and θ is the *angle of arrival* (see Figure 2 on the website).

Suppose that we measure the planewave field at the points $n\Delta$, for some $\Delta > 0$ and $|n| \leq N$. Then we have the data $u(t, n\Delta, 0)$, for $|n| \leq N$. From this data we extract the values

$$\exp(i(\frac{\omega\cos\theta}{c}n\Delta).$$

In practice, there will be planewave fields of varying strengths coming from all directions; let the complex value $f(\theta)$ denote the magnitude and phase of the signal associated with the direction θ. Therefore, the measured data yields the values

$$\int_{-\pi}^{\pi} f(\theta)\exp(i(\frac{\omega\cos\theta}{c}n\Delta)d\theta.$$

We can rewrite this using the variable $k = \frac{\omega\cos\theta}{c}$, with $|k| \leq \frac{\omega}{c}$, and

$$\frac{dk}{d\theta} = -\frac{\omega}{c}\sin\theta = -\frac{1}{c}\sqrt{\omega^2 - c^2k^2}.$$

Therefore, we have the values

$$\frac{1}{c}\int_{-\omega/c}^{\omega/c} f(\arccos(\frac{ck}{\omega}))\exp(ikn\Delta)\sqrt{\omega^2 - c^2k^2}dk.$$

If we choose

$$\Delta = \frac{\pi c}{\omega},$$

then we have the Fourier coefficients c_n of the function

$$g(k) = f(\arccos(\frac{ck}{\omega}))\sqrt{\omega^2 - c^2k^2},$$

for $|n| \leq N$.