Notes on The Laplace Transform and the Ozone Layer

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1 The Laplace Transform and the Ozone Layer

In farfield propagation problems, we often find the measured data to be related to the desired object function by a Fourier transformation. The image reconstruction problem then becomes one of estimating a function from finitely many noisy values of its Fourier transform. In this note we consider an inverse problem involving the Laplace transform. The example is taken from Twomey's book [1].

2 The Laplace Transform

The Laplace transform of the function $f(x)$ defined for $0 \leq x < +\infty$ is the function

$$\mathcal{F}(s) = \int_0^{+\infty} f(x)e^{-sx}dx.$$

3 Scattering of Ultraviolet Radiation

The sun emits ultraviolet (UV) radiation that enters the Earth's atmosphere at an angle θ_0 that depends on the sun's position, and with intensity $I(0)$. Let the x-axis be vertical, with $x = 0$ at the top of the atmosphere and x increasing as we move down to the Earth's surface, at $x = X$. The intensity at x is given by

$$I(x) = I(0)e^{-kx/\cos\theta_0}.$$

Within the ozone layer, the amount of UV radiation scattered in the direction θ is given by

$$S(\theta, \theta_0)I(0)e^{-kx/\cos\theta_0}\Delta p,$$

where $S(\theta, \theta_0)$ is a known parameter, and Δp is the change in the pressure of the ozone within the infinitesimal layer $[x, x + \Delta x]$, and so is proportional to the concentration of ozone within that layer.

4 Measuring the Scattered Intensity

The radiation scattered at the angle θ then travels to the ground, a distance of $X - x$, weakened along the way, and reaches the ground with intensity

$$S(\theta, \theta_0) I(0) e^{-kx/\cos\theta_0} e^{-k(X-x)/\cos\theta} \Delta p.$$

The total scattered intensity at angle θ is then a superposition of the intensities due to scattering at each of the thin layers, and is then

$$S(\theta, \theta_0) I(0) e^{-kX/\cos\theta_0} \int_0^X e^{-x\beta} dp,$$

where

$$\beta = k[\frac{1}{\cos\theta_0} - \frac{1}{\cos\theta}].$$

This superposition of intensity can then be written as

$$S(\theta, \theta_0) I(0) e^{-kX/\cos\theta_0} \int_0^X e^{-x\beta} p'(x) dx.$$

5 The Laplace Transform Data

Using integration by parts, we get

$$\int_0^X e^{-x\beta} p'(x) dx = p(X) e^{-\beta X} - p(0) + \beta \int_0^X e^{-\beta x} p(x) dx.$$

Since $p(0) = 0$ and $p(X)$ can be measured, our data is then the Laplace transform value

$$\int_0^{+\infty} e^{-\beta x} p(x) dx;$$

note that we can replace the upper limit X with $+\infty$ if we extend $p(x)$ as zero beyond $x = X$.

The variable β depends on the two angles θ and θ_0. We can alter θ as we measure and θ_0 changes as the sun moves relative to the earth. In this way we get values of the Laplace transform of $p(x)$ for various values of β. The problem then is to recover $p(x)$ from these values. Because the Laplace transform involves a smoothing of the function $p(x)$, recovering $p(x)$ from its Laplace transform is more ill-conditioned than is the Fourier transform inversion problem.

References

[1] Twomey, S. (1996) *Introduction to the Mathematics of Inversion in Remote Sensing and Indirect Measurement*. New York: Dover Publ.