Incorporating Prior Knowledge in Image Reconstruction through the Choice of the Hilbert Space and Inner Product

Charles Byrne (Charles_Byrne@uml.edu)
Department of Mathematical Sciences
University of Massachusetts at Lowell
Lowell, MA 01854, USA

April 3, 2006

Abstract

The problem is to reconstruct a function $f : R^D \to C$ from finitely many linear functional values. To model the operator that transforms f into the data vector, we need to select an ambient space containing f. Typically, we choose a Hilbert space. The selection of the inner product provides an opportunity to incorporate prior knowledge about f into the reconstructon. The inner product induces a norm and our reconstruction is the function, consistent with the data, for which this norm is minimized. The method is illustrated using Fourier-transform data and prior knowledge about the support of f and about its overall shape.

1 The Basic Problem

We want to reconstruct a function $f : R^D \to C$ from finitely many linear functional values, $g_1, ..., g_N$. Although we may reasonably view f as part of objective reality, once we embed f in a Hilbert space we are imposing theory that, while reasonable, is not given a priori, and is not part of objective reality. As we shall see, the selection of the ambient Hilbert space provides one of the few opportunities we have to incorporate prior knowledge about f, and therefore is an important step in the reconstruction. Because the problem is highly underdetermined, there will be infinitely many reconstructions consistent with the finite data. A common approach to solving such problems is to select the reconstruction with the smallest norm; how we select the norm in the first place is the important step.

1.1 The $L^2(R^D)$ inner product

It is common practice to view f as a member of $L^2(R^D)$ and to model the data values as

$$g_n = \langle f, h^n \rangle_2 = \int f(x)\overline{h^n(x)}dx. \tag{1.1}$$

for known h^n. We then model the problem to be solved as

$$\mathbf{g} = \mathcal{H}f, \tag{1.2}$$

where

$$\mathcal{H}f = (\langle f, h^1 \rangle_2, ..., \langle f, h^N \rangle_2)^T. \tag{1.3}$$

For any continuous linear operator $\mathcal{T}$ on $L^2(R^D)$, the adjoint operator, denoted $\mathcal{T}^\dagger$, is defined by

$$\langle \mathcal{T}f, h \rangle_2 = \langle f, \mathcal{T}^\dagger h \rangle_2.$$

As we change the ambient Hilbert space, or just the inner product, the adjoint operator will change.

1.2 A Class of Inner Products

Let $\mathcal{T}$ be a continuous, linear and invertible operator on $L^2(R^D)$. Define the $\mathcal{T}$ inner product to be

$$\langle f, h \rangle_{\mathcal{T}} = \langle \mathcal{T}^{-1}f, \mathcal{T}^{-1}h \rangle_2. \tag{1.4}$$

We can then use this inner product to define the problem to be solved. We now say that

$$g_n = \langle f, t^n \rangle_{\mathcal{T}}, \tag{1.5}$$

for known functions $t^n(x)$. Using the definition of the $\mathcal{T}$ inner product, we find that

$$g_n = \langle f, h^n \rangle_2 = \langle \mathcal{T}f, \mathcal{T}h^n \rangle_{\mathcal{T}}.$$

The adjoint operator for $\mathcal{T}$, with respect to the $\mathcal{T}$-norm, is denoted $\mathcal{T}^*$, and is defined by

$$\langle \mathcal{T}f, h \rangle_{\mathcal{T}} = \langle f, \mathcal{T}^*h \rangle_{\mathcal{T}}.$$

Therefore,

$$g_n = \langle f, \mathcal{T}^*\mathcal{T}h^n \rangle_{\mathcal{T}}.$$

It is easy to show that $\mathcal{T}^*\mathcal{T} = \mathcal{T}\mathcal{T}^\dagger$. Consequently, we have

$$g_n = \langle f, \mathcal{T}\mathcal{T}^\dagger h^n \rangle_{\mathcal{T}}. \tag{1.6}$$

2 The Minimum-Norm Solutions

Since the reconstruction problem is highly underdetermined, we seek a minimum-norm solution, in whichever norm we are considering. From basic Hilbert space theory, we know that the minimum-norm solution is the function $\hat{f}$ consistent with the data having the algebraic form

$$\hat{f} = \sum_{m=1}^{N} a_m \mathcal{T}\mathcal{T}^{\dagger} h^m. \tag{2.1}$$

Applying the inner product to both sides of Equation (2.1), we get

$$\begin{aligned} g_n &= \langle \hat{f}, \mathcal{T}\mathcal{T}^{\dagger} h^n \rangle_{\mathcal{T}} \\ &= \sum_{m=1}^{N} a_m \langle \mathcal{T}\mathcal{T}^{\dagger} h^m, \mathcal{T}\mathcal{T}^{\dagger} h^n \rangle_{\mathcal{T}}. \end{aligned}$$

Therefore,

$$g_n = \sum_{m=1}^{N} a_m \langle \mathcal{T}^{\dagger} h^m, \mathcal{T}^{\dagger} h^n \rangle_2. \tag{2.2}$$

We solve this system for the a_m and insert them into Equation (2.1) to get our reconstruction. The Gram matrix that appears in Equation (2.2) is positive-definite, but is often ill-conditioned; increasing the main diagonal by a percent or so usually is sufficient regularization.

3 The Case of Fourier-Transform Data

To illustrate these minimum-norm solutions, we consider the case in which the data are values of the Fourier transform of f. Specifically, suppose that

$$g_n = \int f(x) e^{-i\omega_n x} dx,$$

for arbitrary values ω_n. In order to have the functions $e^{i\omega_n x}$ in the Hilbert space, we need to assume that f has bounded support, say $[-A, A]$, and then select the Hilbert space to be $L^2(-A, A)$.

3.1 The $L^2(-\pi, \pi)$ Case

Suppose that the Fourier values are associated with the equi-spaced points $\omega_n = n$. If we do not have prior knowledge of the support of f, we may assume that $f(x) = 0$, for $|x| > \pi$. The minimum-norm solution within $L^2(-\pi, \pi)$ is then

$$\hat{f}(x) = \sum_{n=1}^{N} g_n e^{inx}. \tag{3.1}$$

3.2 The Over-Sampled Case

Suppose that $f(x) = 0$ for $|x| > A$, where $0 < A < \pi$. Then we use $L^2(-A, A)$ as the Hilbert space. For equispaced data at $\omega_n = n$, we have

$$g_n = \int_{-A}^{A} f(x)e^{-inx}dx,$$

so that the minimum-norm solution has the form

$$\hat{f}(x) = \chi_A(x)\sum_{m=1}^{N} a_m e^{imx}, \tag{3.2}$$

with χ_A the characteristic function of the interval $[-A, A]$ and

$$g_n = 2\sum_{m=1}^{N} a_m \frac{\sin A(m-n)}{m-n}.$$

The minimum-norm solution is support-limited to $[-A, A]$ and consistent with the Fourier-transform data.

3.3 Irregularly Spaced Data

For general ω_n the minimum-norm solution within $L^2(-A, A)$ is

$$\hat{f}(x) = \sum_{m=1}^{N} a_m e^{i\omega_m x}, \tag{3.3}$$

with

$$g_n = \sum_{m=1}^{N} a_m \int_{-A}^{A} e^{i(\omega_m - \omega_n)x}dx.$$

In each of these examples we have incorporated prior knowledge of the support of f. Now we consider including information about the shape of the function f.

3.4 Using a Prior Estimate of f

Suppose that $f(x) = 0$ for $|x| > A$ again, and that $p(x)$ satisfies

$$0 < \epsilon \leq p(x) \leq E < +\infty,$$

for all x in $[-A, A]$. Define the operator $\mathcal{T}$ by $(\mathcal{T}f)(x) = \sqrt{p(x)}f(x)$. The $\mathcal{T}$-norm is then

$$\langle f, h\rangle_{\mathcal{T}} = \int_{-A}^{A} f(x)\overline{h(x)}p(x)^{-1}dx.$$

It follows that

$$g_n = \int_{-A}^{A} f(x)p(x)e^{-i\omega_n x}p(x)^{-1}dx,$$

so that the minimum $\mathcal{T}$-norm solution is

$$\hat{f}(x) = \sum_{m=1}^{N} a_m p(x) e^{i\omega_m x} = p(x) \sum_{m=1}^{N} a_m e^{i\omega_m x}, \tag{3.4}$$

where

$$g_n = \sum_{m=1}^{N} a_m \int_{-A}^{A} p(x) e^{i(\omega_m - \omega_n)x} dx.$$

If we have prior knowledge about the support of f, or some idea of its shape, we can incorporate that prior knowledge into the reconstruction through the choice of $p(x)$.

The reconstruction in Equation (3.4) was presented in [2], where it was called the PDFT method. The PDFT was based on an earlier non-iterative version of the Gerchberg-Papoulis bandlimited extrapolation procedure [1]. The PDFT was then applied to image reconstruction problems in [3]. An application of the PDFT was presented in [5]. In [4] we extended the PDFT to a nonlinear version, the indirect PDFT (IPDFT), that generalizes Burg's maximum entropy spectrum estimation method. The PDFT was applied to the phase problem in [6] and in [7] both the PDFT and IPDFT were examined in the context of Wiener filter approximation. More recent work on these topics is discussed in the book [8].

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