Trigonometric Approximation Notes

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To Eileen

Contents

Chapter 1

Complex Numbers

It is standard practice in signal processing to employ complex numbers whenever possible. One of the main reasons for doing this is that it enables us to represent the important sine and cosine functions in terms of complex exponential functions and to replace trigonometric identities with the somewhat simpler rules for the manipulation of exponents.

The complex numbers are the points in the x, y-plane: the complex number $z = (a, b)$ is identified with the point in the plane having $a = Re(z)$, the *real part* of z, for its x-coordinate and $b = Im(z)$, the *imaginary part* of z, for its y-coordinate. We call (a, b) the *rectangular form* of the complex number z. The *conjugate* of the complex number z is $\overline{z} = (a, -b)$. We can also represent z in its polar form: let the *magnitude* of z be $|z| = \sqrt{a^2 + b^2}$ and the *phase angle* of z, denoted $\theta(z)$, be the angle in $[0, 2\pi)$ with $\cos\theta(z) = a/|z|$. Then the *polar form* for z is

$$z = (|z|\cos\theta(z), |z|\sin\theta(z)).$$

Any complex number $z = (a, b)$ for which the imaginary part $Im(z) = b$ is zero is identified with (treated as the same as) its real part $Re(z) = a$; that is, we identify a and $z = (a, 0)$. These real complex numbers lie along the x-axis in the plane, the so-called *real line*. If this were the whole story complex numbers would be unimportant; but they are not. It is the arithmetic associated with complex numbers that makes them important.

We add two complex numbers using their rectangular representations:

$$(a, b) + (c, d) = (a + c, b + d).$$

This is the same formula used to add two-dimensional vectors. We multiply complex numbers more easily when they are in their polar representations: the product of z and w has $|z||w|$ for its magnitude and $\theta(z) + \theta(w)$ modulo 2π for its phase angle. Notice that the complex number $z = (0, 1)$ has

$\theta(z) = \pi/2$ and $|z| = 1$, so $z^2 = (-1, 0)$, which we identify with the real number -1. This tells us that within the realm of complex numbers the real number -1 has a square root, $i = (0, 1)$; note that $-i = (0, -1)$ is also a square root of -1.

To multiply $z = (a, b) = a + ib$ by $w = (c, d) = c + id$ in rectangular form we simply multiply the binomials

$$(a + ib)(c + id) = ac + ibc + iad + i^2bd$$

and recall that $i^2 = -1$ to get

$$zw = (ac - bd, bc + ad).$$

If (a, b) is real, that is, if $b = 0$, then $(a, b)(c, d) = (a, 0)(c, d) = (ac, ad)$, which we also write as $a(c, d)$. Therefore, we can rewrite the polar form for z as

$$z = |z|(\cos\theta(z), \sin\theta(z)) = |z|(\cos\theta(z) + i\sin\theta(z)).$$

We will have yet another way to write the polar form of z when we consider the complex exponential function.

Exercise 1: Derive the formula for dividing one complex number in rectangular form by another (non-zero) one.

Exercise 2: Show that for any two complex numbers z and w we have

$$|zw| \geq \frac{1}{2}(z\overline{w} + \overline{z}w). \tag{1.1}$$

Hint: Write $|zw|$ as $|z\overline{w}|$.

Exercise 3: Show that, for any constant a with $|a| \neq 1$, the function

$$G(z) = \frac{z - \overline{a}}{1 - az}$$

has $|G(z)| = 1$ whenever $|z| = 1$.

Chapter 2

Complex Exponentials

The most important function in signal processing is the complex-valued function of the real variable x defined by

$$h(x) = \cos(x) + i\sin(x). \tag{2.1}$$

For reasons that will become clear shortly, this function is called the *complex exponential function*. Notice that the magnitude of the complex number $h(x)$ is always equal to one, since $\cos^2(x) + \sin^2(x) = 1$ for all real x. Since the functions $\cos(x)$ and $\sin(x)$ are 2π-periodic, that is, $\cos(x+2\pi) = \cos(x)$ and $\sin(x+2\pi) = \sin(x)$ for all x, the complex exponential function $h(x)$ is also 2π-periodic.

In calculus we encounter functions of the form $g(x) = a^x$, where $a > 0$ is an arbitrary constant. These functions are the *exponential* functions, the most well known of which is the function $g(x) = e^x$. Exponential functions are those with the property $g(u+v) = g(u)g(v)$ for every u and v. We show now that the function $h(x)$ in equation (2.1) has this property, so must be an exponential function; that is, $h(x) = c^x$ for some constant c. Since $h(x)$ has complex values, the constant c cannot be a real number, however.

Calculating $h(u)h(v)$ we find

$$h(u)h(v) = (\cos(u)\cos(v) - \sin(u)\sin(v)) + i(\cos(u)\sin(v) + \sin(u)\cos(v))$$

$$= \cos(u+v) + i\sin(u+v) = h(u+v).$$

So $h(x)$ is an exponential function; $h(x) = c^x$ for some complex constant c. Inserting $x = 1$ we find that c is

$$c = \cos(1) + i\sin(1).$$

Let's try to find another way to express c.

Recall from calculus that for exponential functions $g(x) = a^x$ with $a > 0$ the derivative $g'(x)$ is

$$g'(x) = a^x \ln(a) = g(x)\ln(a).$$

Since

$$h'(x) = -\sin(x) + i\cos(x) = i(\cos(x) + i\sin(x)) = ih(x)$$

we conjecture that $\ln(c) = i$; but what does this mean?

For $a > 0$ we know that $b = \ln(a)$ means that $a = e^b$. Therefore, we say that $\ln(c) = i$ means $c = e^i$; but what does it mean to take e to a complex power? To define e^i we turn to the Taylor series representation for the exponential function $g(x) = e^x$, defined for real x:

$$e^x = 1 + x + x^2/2! + x^3/3! + ...$$

Inserting i in place of x and using the fact that $i^2 = -1$, we find that

$$e^i = (1 - 1/2! + 1/4! - ...) + i(1 - 1/3! + 1/5! - ...);$$

note that the two series are the Taylor series for $\cos(1)$ and $\sin(1)$, respectively, so $e^i = \cos(1) + i\sin(1)$. Then the complex exponential function in equation (2.1) is

$$h(x) = (e^i)^x = e^{ix}.$$

Inserting $x = \pi$ we get

$$h(\pi) = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

or

$$e^{i\pi} + 1 = 0,$$

which is the remarkable relation discovered by Euler that combines the five most important constants in mathematics, e, π, i, 1 and 0, in a single equation.

Note that $e^{2\pi i} = e^{0i} = e^0 = 1$, so

$$e^{(2\pi+x)i} = e^{2\pi i}e^{ix} = e^{ix}$$

for all x.

We know from calculus what e^x means for real x and now we also know what e^{ix} means. Using these we can define e^z for any complex number $z = a + ib$ by $e^z = e^{a+ib} = e^a e^{ib}$.

We know from calculus how to define $\ln(x)$ for $x > 0$ and we have just defined $\ln(c) = i$ to mean $c = e^i$. But we could also say that $\ln(c) = i(1 + 2\pi k)$ for any integer k; that is, the periodicity of the complex exponential function forces the function $\ln(x)$ to be multivalued.

For any nonzero complex number $z = |z|e^{i\theta(z)}$ we have

$$\ln(z) = \ln(|z|) + \ln(e^{i\theta(z)}) = \ln(|z|) + i(\theta(z) + 2\pi k),$$

for any integer k. If $z = a > 0$ then $\theta(z) = 0$ and $\ln(z) = \ln(a) + i(k\pi)$ for any even integer k; in calculus class we just take the value associated with $k = 0$. If $z = a < 0$ then $\theta(z) = \pi$ and $\ln(z) = \ln(-a) + i(k\pi)$ for any odd integer k. So we can define the logarithm of a negative number; it just turns out not to be a real number. If $z = ib$ with $b > 0$, then $\theta(z) = \frac{\pi}{2}$ and $\ln(z) = \ln(b) + i(\frac{\pi}{2} + 2\pi k)$, for any integer k; if $z = ib$ with $b < 0$ then $\theta(z) = \frac{3\pi}{2}$ and $\ln(z) = \ln(-b) + i(\frac{3\pi}{2} + 2\pi k)$ for any integer k.

Adding $e^{-ix} = \cos(x) - i\sin(x)$ to e^{ix} given by equation (2.1) we get

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix});$$

subtracting, we obtain

$$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

These formulas allow us to extend the definition of cos and sin to complex arguments z:

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$$

and

$$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

In signal processing the complex exponential function is often used to describe functions of time that exhibit periodic behavior:

$$h(\omega t + \theta) = e^{i(\omega t + \theta)} = \cos(\omega t + \theta) + i\sin(\omega t + \theta),$$

where the *frequency* ω and *phase angle* θ are real constants, and t denotes time. We can alter the magnitude by multiplying $h(\omega t + \theta)$ by a positive constant $|A|$, called the *amplitude*, to get $|A|h(\omega t + \theta)$. More generally, we can combine the amplitude and the phase, writing

$$|A|h(\omega t + \theta) = |A|e^{i\theta}e^{i\omega t} = Ae^{i\omega t},$$

where A is the complex amplitude $A = |A|e^{i\theta}$. Many of the functions encountered in signal processing can be modeled as linear combinations of such complex exponential functions or *sinusoids*, as they are often called.

Exercise 1: Show that if $\sin\frac{x}{2} \neq 0$ then

$$E_M(x) = \sum_{m=1}^{M} e^{imx} = e^{ix(\frac{M+1}{2})}\frac{\sin(Mx/2)}{\sin(x/2)}. \tag{2.2}$$

Hint: Note that $E_M(x)$ is the geometric progression

$$E_M(x) = e^{ix} + (e^{ix})^2 + (e^{ix})^3 + ... + (e^{ix})^M = e^{ix}(1 - e^{iMx})/(1 - e^{ix}).$$

Now use the fact that, for any t, we have

$$1 - e^{it} = e^{it/2}(e^{-it/2} - e^{it/2}) = e^{it/2}(-2i)\sin(t/2).$$

Exercise 2: The *Dirichlet kernel* of size M is defined as

$$D_M(x) = \sum\nolimits_{m=-M}^{M} e^{imx}.$$

Use equation (2.2) to obtain the closed-form expression

$$D_M(x) = \frac{\sin((M + \frac{1}{2})x)}{\sin(\frac{x}{2})};$$

note that $D_M(x)$ is real-valued.
Hint: Reduce the problem to that of Exercise 1 by factoring appropriately.

Exercise 3: Use the result in equation (2.2) to obtain the closed-form expressions

$$\sum\nolimits_{m=N}^{M} \cos mx = \cos(\frac{M+N}{2}x)\frac{\sin(\frac{M-N+1}{2}x)}{\sin\frac{x}{2}}$$

and

$$\sum\nolimits_{m=N}^{M} \sin mx = \sin(\frac{M+N}{2}x)\frac{\sin(\frac{M-N+1}{2}x)}{\sin\frac{x}{2}}.$$

Hint: Recall that $\cos mx$ and $\sin mx$ are the real and imaginary parts of e^{imx}.

Exercise 4: Graph the function $E_M(x)$ for various values of M.

We note in passing that the function $E_M(x)$ equals M for $x = 0$ and equals zero for the first time at $x = 2\pi/M$. This means that the *main lobe* of $E_M(x)$, the inverted parabola-like portion of the graph centered at $x = 0$, crosses the x-axis at $x = 2\pi/M$ and $x = -2\pi/M$, so its height is M and its width is $4\pi/M$. As M grows larger the main lobe of $E_M(x)$ gets higher and thinner.

Chapter 3

Convolution and the Vector DFT

Convolution is an important concept in signal processing and occurs in several distinct contexts. In this chapter we shall discuss *non-periodic convolution* and *periodic convolution* of vectors. Later we shall consider the convolution of infinite sequences and of functions of a continuous variable. The reader may recall an earlier encounter with convolution in a course on differential equations. The simplest example of convolution is the non-periodic convolution of finite vectors.

Non-periodic convolution:

Recall the algebra problem of multiplying one polynomial by another. Suppose

$$A(x) = a_0 + a_1 x + ... + a_M x^M$$

and

$$B(x) = b_0 + b_1 x + ... + b_N x^N.$$

Let $C(x) = A(x)B(x)$. With

$$C(x) = c_0 + c_1 x + ... + c_{M+N} x^{M+N},$$

each of the coefficients c_j, $j = 0, ..., M+N$, can be expressed in terms of the a_m and b_n (an easy exercise!). The vector $c = (c_0, ..., c_{M+N})$ is called the *non-periodic convolution* of the vectors $a = (a_0, ..., a_M)$ and $b = (b_0, ..., b_N)$. Non-periodic convolution can be viewed as a particular case of periodic convolution, as we see next.

The DFT and the vector DFT:

As we just discussed, non-periodic convolution is another way of looking at the multiplication of two polynomials. This relationship between convolution on the one hand and multiplication on the other is a fundamental aspect of convolution, whenever it occurs. Whenever we have a convolution we should ask what related mathematical objects are being multiplied. We ask this question now with regard to periodic convolution; the answer turns out to be the *vector discrete Fourier transform*.

Given the N by 1 vector $\mathbf{f}$ with complex entries $f_0, f_1, ..., f_{N-1}$ define the *discrete Fourier transform* (DFT) of $\mathbf{f}$ to be the function $DFT_{\mathbf{f}}(\omega)$, defined for ω in $[0, 2\pi)$, by

$$DFT_{\mathbf{f}}(\omega) = \sum_{n=0}^{N-1} f_n e^{in\omega}.$$

The terminology can be confusing, since the expression 'discrete Fourier transform' is often used to describe several slightly different mathematical objects.

For example, in the exercise that follows we are interested solely in the values $F_k = DFT_{\mathbf{f}}(2\pi k/N)$, for $k = 0, 1, ..., N-1$. In this case the DFT of the vector $\mathbf{f}$ often means simply the vector $\mathbf{F}$ whose entries are the complex numbers F_k, for $k = 0, ..., N-1$; for the moment let us call this the *vector DFT* of $\mathbf{f}$ and write $\mathbf{F} = vDFT_{\mathbf{f}}$. The point of Exercise 1 is to show how to use the vector DFT to perform the *periodic convolution* operation.

In some instances the numbers f_n are obtained by evaluating a function $f(x)$ at some finite number of points x_n; that is, $f_n = f(x_n)$, for $n = 0, ..., N-1$. As we shall see later, if the x_n are equispaced, the DFT provides an approximation of the Fourier transform of the function $f(x)$. Since the Fourier transform is another function of a continuous variable, and not a vector, it is appropriate, then, to view the DFT also as such a function. Since the practice is to use the term DFT to mean slightly different things in different contexts, we adopt that practice here. The reader will have to infer the precise meaning of DFT from the context.

Periodic convolution:

Given the N by 1 vectors $\mathbf{f}$ and $\mathbf{d}$ with complex entries f_n and d_n, respectively, we define a third N by 1 vector $\mathbf{f} * \mathbf{d}$, the *periodic convolution* of $\mathbf{f}$ and $\mathbf{d}$, to have the entries

$$(\mathbf{f} * \mathbf{d})_n = f_0 d_n + f_1 d_{n-1} + ... + f_n d_0 + f_{n+1} d_{N-1} + ... + f_{N-1} d_{n+1}.$$

Periodic convolution is illustrated in Figure 3.1. The first exercise relates the periodic convolution to the vector DFT.

Exercise 1: Let $\mathbf{F} = vDFT_{\mathbf{f}}$ and $\mathbf{D} = vDFT_{\mathbf{d}}$. Define a third vector $\mathbf{E}$ having for its k-th entry $E_k = F_k D_k$, for $k = 0, ..., N-1$. Show that $\mathbf{E}$ is the vDFT of the vector $\mathbf{f} * \mathbf{d}$.

The vector $vDFT_{\mathbf{f}}$ can be obtained from the vector $\mathbf{f}$ by means of matrix multiplcation by a certain matrix G, called the *DFT matrix*. The matrix G has an inverse that is easily computed and can be used to go from $\mathbf{F} = vDFT_{\mathbf{f}}$ back to the original $\mathbf{f}$. The details are in Exercise 2.

Exercise 2: Let G be the N by N matrix whose entries are $G_{jk} = e^{i(j-1)(k-1)2\pi/N}$. The matrix G is sometimes called the *DFT matrix*. Show that the inverse of G is $G^{-1} = \frac{1}{N}G^{\dagger}$, where $G^{\dagger}$ is the conjugate transpose of the matrix G. Then $\mathbf{f} * \mathbf{d} = G^{-1}\mathbf{E} = \frac{1}{N}G^{\dagger}\mathbf{E}$.

As we mentioned above, nonperiodic convolution is really a special case of periodic convolution. Extend the $M+1$ by 1 vector a to an $M+N+1$ by 1 vector by appending N zero entries; similarly, extend the vector b to an $M+N+1$ by 1 vector by appending zeros. The vector c is now the periodic convolution of these extended vectors. Therefore, since we have an efficient algorithm for performing periodic convolution, namely the Fast Fourier Transform algorithm (FFT), we have a fast way to do the periodic (and thereby nonperiodic) convolution and polynomial multiplication.

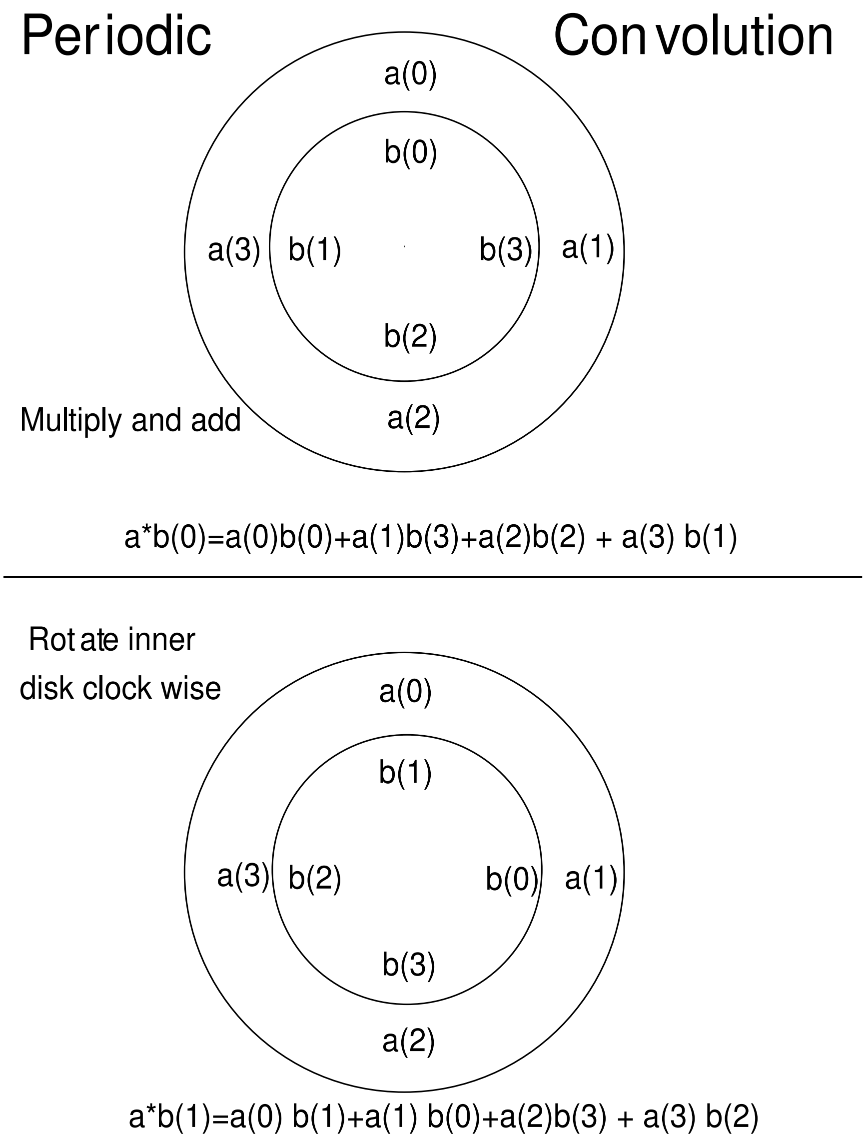

Figure 3.1: Periodic convolution of vectors $a = (a(0), a(1), a(2), a(3))$ and $b = (b(0), b(1), b(2), b(3))$.

Chapter 4

Fourier Transforms and Fourier Series

In a previous chapter we studied the problem of isolating the individual complex exponential components of the signal function $s(t)$, given the data vector $\mathbf{d}$ with entries $s(m\Delta)$, $m = 1, ..., M$, where $s(t)$ is

$$s(t) = \sum_{n=1}^{N} A_n e^{i\omega_n t};$$

we assume that $|\omega_n| < \pi/\Delta$. The second approach we considered involved calculating the function

$$DFT_{\mathbf{d}}(\omega) = \sum_{m=1}^{M} s(m\Delta) e^{-i\omega m\Delta}$$

for $|\omega| < \pi/\Delta$. This sum is an example of a (finite) Fourier series. As we just saw, we can extend the concept of Fourier series to include infinite sums. In fact, we can generalize to summing over a continuous variable, using integrals in place of summation; this is what is done in the definition of the Fourier transform.

The Fourier transform:

In our discussion of linear filtering we saw that if f is a finite vector $\mathbf{f} = (f_1, ..., f_M)^T$ or an infinite sequence $f = \{f_m\}_{m=-\infty}^{+\infty}$ then it is convenient to consider the function $F(\omega)$ defined for $|\omega| \leq \pi$ by the finite or infinite Fourier series expression

$$F(\omega) = \sum f_m e^{im\omega}.$$

If $f(x)$ is a function of the real variable x, we can associate with f the function $F(\omega)$, the *Fourier transform* (FT) of $f(x)$, defined for all real ω

by

$$F(\omega) = \int f(x)e^{ix\omega}dx. \tag{4.1}$$

Once we have $F(\omega)$ we can recover $f(x)$ as the *inverse Fourier transform* (IFT) of $F(\omega)$:

$$f(x) = \int F(\omega)e^{-ix\omega}d\omega/2\pi. \tag{4.2}$$

We say then that the functions f and F form a Fourier transform pair. It may happen that one or both of the integrals above will fail to be defined in the usual way and will be interpreted as the principal value of the integral [97].

Note that the definitions of the FT and IFT just given may differ slightly from the ones found elsewhere; our definitions are those of Bochner and Chandrasekharan [18]. The differences are minor and involve only the placement of the quantity 2π and of the minus sign in the exponent. One sometimes sees the FT of the function f denoted $\hat{f}$; here we shall reserve the symbol $\hat{f}$ for estimates of the function f.

As an example of a Fourier transform pair let $F(\omega)$ be the function $\chi_\Omega(\omega)$ that equals one for $|\omega| \leq \Omega$ and is zero otherwise. Then the inverse Fourier transform of $\chi_\Omega(\omega)$ is

$$f(x) = \int_{-\Omega}^{\Omega} e^{-i\omega x}d\omega/2\pi = \frac{\sin(\Omega x)}{\pi x}.$$

The function $\frac{\sin(x)}{x}$ is called the *sinc* function, $\operatorname{sinc}(x)$.

Fourier series:

If there is a positive Ω such that the Fourier transform $F(\omega)$ of the function $f(x)$ is zero for $|\omega| > \Omega$ then the function $f(x)$ is said to be Ω-*bandlimited* and $F(\omega)$ has *bandwidth* Ω; in this case the function $F(\omega)$ can be written, on the interval $[-\Omega, \Omega]$, as an infinite discrete sum of complex exponentials. For $|\omega| \leq \Omega$ we have

$$F(\omega) = \sum\nolimits_{n=-\infty}^{+\infty} f_n e^{in\omega\frac{\pi}{\Omega}}. \tag{4.3}$$

We determine the coefficients f_n in much the same way as in earlier discussions.

We know that the integral

$$\int_{-\Omega}^{\Omega} e^{i(n-m)\omega\frac{\pi}{\Omega}}d\omega$$

equals zero if $m \neq n$ and equals 2Ω for $m = n$. Therefore,

$$f_m = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\omega) e^{-im\omega \frac{\pi}{\Omega}} d\omega \tag{4.4}$$

for each integer m. If we wish, we can also write the coefficient f_m in terms of the inverse Fourier transform $f(x)$ of the function $F(\omega)$: the right side of equation (4.4) also equals $\frac{\pi}{\Omega} f(m\frac{\pi}{\Omega})$, from which we conclude that $f_m = \frac{\pi}{\Omega} f(m\frac{\pi}{\Omega})$.

The Shannon Sampling Theorem: Now that we have found the coefficients of the Fourier series for $F(\omega)$ we can write

$$F(\omega) = \frac{\pi}{\Omega} \sum_{n=-\infty}^{\infty} f(n\frac{\pi}{\Omega}) e^{in\omega \frac{\pi}{\Omega}} \tag{4.5}$$

for $|\omega| \leq \Omega$. We apply the formula in equation (4.2) to get

$$f(x) = \sum_{n=-\infty}^{\infty} f(n\frac{\pi}{\Omega}) \frac{\sin(\Omega x - n\pi)}{\Omega x - n\pi}. \tag{4.6}$$

This is the famous *Shannon sampling theorem*, which tells us that if $F(\omega)$ is zero outside $[-\Omega, \Omega]$, then $f(x)$ is completely determined by the infinite sequence of values $\{f(n\frac{\pi}{\Omega})\}_{n=-\infty}^{+\infty}$. If $F(\omega)$ is continuous and $F(-\Omega) = F(\Omega)$ then $F(\omega)$ has a continuous periodic extension to all of the real line. Then the Fourier series in equation (4.3) converges to $F(\omega)$ for every ω at which the function $F(\omega)$ has a left and right derivative. In general, if $F(-\Omega) \neq F(\Omega)$, or if $F(\omega)$ is discontinuous for some ω in $(-\Omega, \Omega)$, the series will still converge, but to the average of the one-sided limits $F(\omega+0)$ and $F(\omega - 0)$, again, provided that $F(\omega)$ has one-sided derivatives at that point. If

$$\int_{-\Omega}^{\Omega} |F(\omega)|^2 d\omega < \infty$$

then

$$\sum_{n=-\infty}^{+\infty} |f(n\frac{\pi}{\Omega})|^2 < \infty$$

and the series in equation (4.6) converges to $f(x)$ in the L^2 sense. If, in addition, we have

$$\sum_{n=-\infty}^{+\infty} |f(n\frac{\pi}{\Omega})| < \infty,$$

then the series converges uniformly to $f(x)$ for x on the real line. There are many books that can be consulted for details concerning convergence of Fourier series, such as [16] and [97].

Let $f = \{f_m\}$ and $g = \{g_m\}$ be the sequences of Fourier coeffcients for the functions $F(\omega)$ and $G(\omega)$, respectively, defined on the interval $[-\pi, \pi]$; that is

$$F(\omega) = \sum_{m=-\infty}^{\infty} f_m e^{im\omega}, |\omega| \leq \pi.$$

Exercise 1: Use the orthogonality of the functions $e^{im\omega}$ on $[-\pi, \pi]$ to establish *Parseval's equation*:

$$\langle f, g \rangle = \sum_{m=-\infty}^{\infty} f_m \overline{g_m} = \int_{-\pi}^{\pi} F(\omega)\overline{G(\omega)} d\omega/2\pi,$$

from which it follows that

$$\langle f, f \rangle = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega/2\pi.$$

Similar results hold for the Fourier transform, as we shall see in the next chapter.

Exercise 2: Let $f(x)$ be defined for all real x and let $F(\omega)$ be its FT. Let

$$g(x) = \sum_{k=-\infty}^{\infty} f(x + 2\pi k),$$

assuming the sum exists. Show that g is a 2π -periodic function. Compute its Fourier series and use it to derive the *Poisson summation formula*:

$$\sum_{k=-\infty}^{\infty} f(2\pi k) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n).$$

In certain applications our main interest is the function $f(x)$, for which we have finitely many (usually noisy) values. For example, x may be the time variable t and $f(t)$ may be a short segment of spoken speech that we wish to analyze. We model $f(t)$ as a finite, infinite discrete or continuous sum of complex exponentials, that is, as a Fourier series or Fourier transform, in order to process the data, to remove the noise, to compress the data and to identify the parameters.

In remote sensing applications (such as radar, sonar, tomography), on the other hand, we have again noisy values of $f(x)$, but it is not $f(x)$ that interests us. Instead, we are interested in $F(\omega)$, the Fourier transform of $f(x)$ or the sequence F_n of the complex Fourier coeffcients of $f(x)$, if $f(x) = 0$ outside some finite interval. We cannot measure these quantities directly, so we must content ourselves with estimating them from our measurements of $f(x)$.

In yet a third class of applications, such as linear filtering, we are concerned with constructing a digital procedure for performing certain operations on any signal we might receive as input. In such cases our goal is to construct the sequence g_n for which the associated Fourier series $G(\omega)$ will have a desired shape. For example, we may want the filter to eliminate all complex exponential components of the input signal whose frequency is not in the interval $[-\Omega, \Omega]$. Then we would want $G(\omega)$ to be one for ω within this interval and zero outside. To achieve this we would take the sequence g_n to be

$$g_n = \frac{\sin(\Omega n)}{\pi n}.$$

In these applications there is no $f(x)$ to be analyzed nor $F(\omega)$ to be estimated.

Chapter 5

Fourier Series and Analytic Functions

We first encounter infinite series expansions for functions in calculus, when we study Maclaurin and Taylor series. Fourier series are usually first met in a much different context, such as partial differential equations and boundary value problems. Laurent expansions come later, when we study functions of a complex variable. There are, nevertheless, important connections among these different types of infinite series expansions, which provide the subject for this chapter.

Suppose that $f(z)$ is analytic in an annulus containing the unit circle $C = \{z \,|\, |z| = 1\}$. Then $f(z)$ has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} f_n z^n$$

valid for z within that annulus. Substituting $z = e^{i\theta}$ we get $f(\theta)$, defined for θ in the interval $[-\pi, \pi]$ by

$$f(\theta) = f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta};$$

here the Fourier series for $f(\theta)$ is derived from the Laurent series for the analytic function $f(z)$. If $f(z)$ is actually analytic in $(1+\epsilon)D$, where $D = \{z| \, |z| < 1\}$ is the open unit disk, then $f(z)$ has a Taylor series expansion and the Fourier series for $f(\theta)$ contains only terms corresponding to nonnegative n.

As an example, consider the rational function

$$f(z) = \frac{1}{z - \frac{1}{2}} - \frac{1}{z-3} = -\frac{5}{2}/(z - \frac{1}{2})(z-3).$$

In an annulus containing the unit circle this function has the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{-1} 2^{n+1} z^n + \sum_{n=0}^{\infty} (\frac{1}{3})^{n+1} z^n;$$

replacing z with $e^{i\theta}$ we obtain the Fourier series for the function $f(\theta) = f(e^{i\theta})$ defined for θ in the interval $[-\pi, \pi]$.

The function $F(z) = 1/f(z)$ is analytic for all complex z, but because it has a root inside the unit circle, its reciprocal, $f(z)$, is not analytic in a disk containing the unit circle. Consequently, the Fourier series for $f(\theta)$ is doubly infinite. We saw in the chapter on complex varables that the function $G(z) = \frac{z-\overline{a}}{1-az}$ has $|G(e^{i\theta})| = 1$. With $a = 2$ and $H(z) = F(z)G(z)$ we have

$$H(z) = \frac{1}{5}(z-3)(z-2)$$

and its reciprocal has the form

$$1/H(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Because

$$G(e^{i\theta})/H(e^{i\theta}) = 1/F(e^{i\theta})$$

it follows that

$$|1/H(e^{i\theta})| = |1/F(e^{i\theta})| = |f(\theta)|$$

and so

$$|f(\theta)| = |\sum_{n=0}^{\infty} a_n e^{in\theta}|.$$

Multiplication by $G(z)$ permits us to move a root from inside C to outside C without altering the magnitude of the function's values on C.

The relationships that obtain between functions defined on C and functions analytic (or harmonic) in D form the core of *harmonic analysis* [114]. The factorization $F(z) = H(z)/G(z)$ above is a special case of the *inner-outer factorization* for functions in Hardy spaces; the function $H(z)$ is an *outer function* and the functions $G(z)$ and $1/G(z)$ are *inner functions*.

Instead of starting with an analytic function and restricting it to the unit circle, we often begin with a function $f(e^{i\theta})$ defined on the unit circle, or, equivalently, a function of the form $f(\theta)$ for θ in $[-\pi, \pi]$, and wish to view this function as the restriction to the unit circle of a function that is analytic in a region containing the unit circle. One application of this idea is the Fejér-Riesz factorization theorem.

Theorem 5.1 *Let $h(\theta)$ be a finite trigonometric polynomial*

$$h(\theta) = \sum_{n=-N}^{N} h_n e^{in\theta}$$

such that $h(\theta) \geq 0$ for all θ in the interval $[-\pi, \pi]$. Then there is

$$y(\theta) = \sum_{n=0}^{N} y_n e^{in\theta}$$

with $h(\theta) = |y(\theta)|^2$. The function $y(z)$ is unique if we require, in addition, that all its roots be outside D.

To prove this theorem we consider the function

$$h(z) = \sum_{n=-N}^{N} h_n z^n,$$

which is analytic in an annulus containing the unit circle, with $h(e^{i\theta}) = h(\theta)$. The rest of the proof is contained in the following exercise.

Exercise 1: Use the fact that $h_{-n} = \overline{h}_n$ to show that z_j is a root of $h(z)$ if and only if $1/\overline{z}_j$ is also a root. From the nonnegativity of $h(e^{i\theta})$ conclude that if $h(z)$ has a root on the unit circle then it has even multiplicity. Take $y(z)$ to be proportional to the product of factors $z - z_j$ for all the z_j outside D; for roots on C include them with half their multiplicities.

The Fejér-Riesz theorem is used in the derivation of Burg's maximum entropy method for spectrum estimation. The problem there is to estimate a function $R(\theta) > 0$ knowing only the values

$$r_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} R(\theta) e^{-in\theta} d\theta,$$

for $|n| \leq N$. The approach is to estimate $R(\theta)$ by the function $S(\theta) > 0$ that maximizes the so-called Burg entropy, $\int_{-\pi}^{\pi} \log S(\theta) d\theta$, subject to the data constraints.

The Euler-Lagrange equation from the calculus of variations allows us to conclude that $S(\theta)$ has the form

$$S(\theta) = 1/ \sum_{n=-N}^{N} h_n e^{in\theta}.$$

The function

$$h(\theta) = \sum_{n=-N}^{N} h_n e^{in\theta}$$

is nonnegative, so, by the Fejér-Riesz theorem, it factors as $h(\theta) = |y(\theta)|^2$. We then have $S(\theta)\overline{y(\theta)} = 1/y(\theta)$. Since all the roots of $y(z)$ lie outside D and none are on C, the function $1/y(z)$ is analytic in a region containing C and D so it has a Taylor series expansion in that region. Restricting this Taylor series to C we obtain a one-sided Fourier series having zero terms for the negative indices.

Exercise 2: Show that the coefficients y_n in $y(z)$ satisfy a system of linear equations whose coefficients are the r_n.

Hint: Compare the coefficients of the terms on both sides of the equation $S(\theta)\overline{y}(\theta) = 1/y(\theta)$ that correspond to negative indices.

The Hilbert transform for sequences: If $g(\omega)$ has the Fourier series expansion

$$g(\omega) = \sum_{n=-\infty}^{\infty} g_n e^{-in\omega},$$

the *conjugate Fourier series* [125] is

$$h(\omega) = \sum_{n=-\infty}^{\infty} (-i\,\mathrm{sgn}(n)) g_n e^{-in\omega}.$$

Then

$$f(\omega) = g(\omega) + ih(\omega) = g_0 + 2\sum_{n=1}^{\infty} g_n e^{in\omega}$$

is a one-sided Fourier series. In harmonic analysis the sequence $\{h_n\}$ is said to be the *conjugate* of the sequence $\{g_n\}$; in signal processing it is called its *Hilbert transform.* As we shall see in a subsequent chapter, the Hilbert transform occurs in several different contexts.

Chapter 6

More on the Fourier Transform

We begin with exercises that treat basic properties of the FT and then introduce several examples of Fourier transform pairs.

Exercise 1: Let $F(\omega)$ be the FT of the function $f(x)$. Use the definitions of the FT and IFT given in equations (4.1) and (4.2) to establish the following basic properties of the Fourier transform operation:

Symmetry: The FT of the function $F(x)$ is $2\pi f(-\omega)$. For example, the FT of the function $f(x) = \frac{\sin(\Omega x)}{\pi x}$ is $\chi_\Omega(\omega)$, so the FT of $g(x) = \chi_\Omega(x)$ is $G(\omega) = 2\pi \frac{\sin(\Omega\omega)}{\pi\omega}$.

Conjugation: The FT of $\overline{f(x)}$ is $\overline{F(-\omega)}$.

Scaling: The FT of $f(ax)$ is $\frac{1}{|a|}F(\frac{\omega}{a})$ for any nonzero constant a.

Shifting: The FT of $f(x-a)$ is $e^{-ia\omega}F(\omega)$.

Modulation: The FT of $f(x)\cos(\omega_0 x)$ is $\frac{1}{2}[F(\omega+\omega_0)+F(\omega-\omega_0)]$.

Differentiation: The FT of the n-th derivative, $f^{(n)}(x)$ is $(-i\omega)^n F(\omega)$. The IFT of $F^{(n)}(\omega)$ is $(ix)^n f(x)$.

Convolution in x: Let f, F, g, G and h, H be FT pairs, with

$$h(x) = \int f(y)g(x-y)dy,$$

so that $h(x) = (f * g)(x)$ is the convolution of $f(x)$ and $g(x)$. Then $H(\omega) = F(\omega)G(\omega)$. For example, if we take $g(x) = \overline{f(-x)}$, then

$$h(x) = \int f(x+y)\overline{f(y)}dy = \int f(y)\overline{f(y-x)}dy = r_f(x)$$

is the *autocorrelation function* associated with $f(x)$ and

$$H(\omega) = |F(\omega)|^2 = R_f(\omega) \geq 0$$

is the *power spectrum* of $f(x)$.

Convolution in ω: Let f, F, g, G and h, H be FT pairs, with $h(x) = f(x)g(x)$. Then $H(\omega) = \frac{1}{2\pi}(F * G)(\omega)$.

Exercise 2: Show that the Fourier transform of $f(x) = e^{-\alpha^2 x^2}$ is $F(\omega) = \frac{\sqrt{\pi}}{\alpha}e^{-(\frac{\omega}{2\alpha})^2}$. Hint: Calculate the derivative $F'(\omega)$ by differentiating under the integral sign in the definition of F and integrating by parts. Then solve the resulting differential equation.

Let $u(x)$ be the *Heaviside function* that is $+1$ if $x \geq 0$ and 0 otherwise. Let $\chi_X(x)$ be the *characteristic function* of the interval $[-X, X]$ that is $+1$ for x in $[-X, X]$ and 0 otherwise. Let $\operatorname{sgn}(x)$ be the *sign function* that is $+1$ if $x > 0$, -1 if $x < 0$ and zero for $x = 0$.

Exercise 3: Show that the FT of the function $f(x) = u(x)e^{-ax}$ is $F(\omega) = \frac{1}{a - i\omega}$, for every positive constant a.

Exercise 4: Show that the FT of $f(x) = \chi_X(x)$ is $F(\omega) = 2\frac{\sin(X\omega)}{\omega}$.

Exercise 5: Show that the IFT of the function $F(\omega) = 2i/\omega$ is $f(x) = \operatorname{sgn}(x)$.

Hints: write the formula for the inverse Fourier transform of $F(\omega)$ as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2i}{\omega}\cos\omega x d\omega - \frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{2i}{\omega}\sin\omega x d\omega$$

which reduces to

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\omega}\sin\omega x d\omega,$$

since the integrand of the first integral is odd. For $x > 0$ consider the Fourier transform of the function $\chi_x(t)$. For $x < 0$ perform the change of variables $u = -x$.

We saw earlier that the $F(\omega) = \chi_\Omega(\omega)$ has for its inverse Fourier transform the function $f(x) = \frac{\sin \Omega x}{\pi x}$; note that $f(0) = \frac{\Omega}{\pi}$ and $f(x) = 0$ for the first time when $\Omega x = \pi$ or $x = \frac{\pi}{\Omega}$. For any Ω-bandlimited function $g(x)$ we have $G(\omega) = G(\omega)\chi_\Omega(\omega)$, so that, for any x_0, we have

$$g(x_0) = \int_{-\infty}^{\infty} g(x) \frac{\sin \Omega(x - x_0)}{\pi(x - x_0)} dx.$$

We describe this by saying that the function $f(x) = \frac{\sin \Omega x}{\pi x}$ has the *sifting property* for all Ω-bandlimited functions $g(x)$.

As Ω grows larger, $f(0)$ approaches $+\infty$, while $f(x)$ goes to zero for $x \neq 0$. The limit is therefore not a function; it is a *generalized function* called *the Dirac delta function at zero*, denoted $\delta(x)$. For this reason the function $f(x) = \frac{\sin \Omega x}{\pi x}$ is called an *approximate delta function.* The FT of $\delta(x)$ is the function $F(\omega) = 1$ for all ω. The Dirac delta function $\delta(x)$ enjoys the *sifting property* for all $g(x)$; that is,

$$g(x_0) = \int_{-\infty}^{\infty} g(x)\delta(x - x_0) dx.$$

It follows from the sifting and shifting properties that the FT of $\delta(x - x_0)$ is the function $e^{ix_0\omega}$.

The formula for the inverse FT nows says

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\omega} d\omega. \tag{6.1}$$

If we try to make sense of this integral according to the rules of calculus we get stuck quickly. The problem is that the integral formula doesn't mean quite what it does ordinarily and the $\delta(x)$ is not really a function, but an operator on functions; it is sometimes called a *distribution.* The Dirac deltas are mathematical fictions, not in the bad sense of being lies or fakes, but in the sense of being made up for some purpose. They provide helpful descriptions of impulsive forces, probability densities in which a discrete point has nonzero probability, or, in array processing, objects far enough away to be viewed as occupying a discrete point in space.

We shall treat the relationship expressed by equation (6.1) as a formal statement, rather than attempt to explain the use of the integral in what is surely an unconventional manner. Nevertheless, it is possible to motivate this relationship by proving that, for any $x \neq 0$,

$$\int_{-\infty}^{\infty} e^{-ix\omega} d\omega = 0.$$

Assume, for convenience, that $x > 0$. Notice first that we can write

$$\int_{-\infty}^{\infty} e^{-ix\omega} d\omega = \sum_{k=-\infty}^{\infty} \int_{\frac{2\pi}{x}k}^{\frac{2\pi}{x}(k+1)} e^{-ix\omega} d\omega.$$

Since

$$e^{-ix\omega} = e^{-ix(\omega + \frac{2\pi}{x})}$$

we can write

$$\int_{\frac{2\pi}{x}k}^{\frac{2\pi}{x}(k+1)} e^{-ix\omega} d\omega = \int_{-\frac{\pi}{x}}^{\frac{\pi}{x}} e^{-ix\omega} d\omega$$

$$= \int_0^{\frac{\pi}{x}} [e^{-ix\omega} + e^{-ix(\omega - \frac{\pi}{x})}] d\omega$$

$$= \frac{1}{x} \int_0^{\pi} [e^{-i\omega}(1 + e^{i\pi})] d\omega$$

$$= \frac{1}{x}(1 + e^{i\pi}) \int_0^{\pi} e^{-i\omega} d\omega = 0.$$

Clearly, when $x = 0$ the integrand is one for all ω, which leads to the delta function supported at zero.

If we move the discussion into the ω domain and define the Dirac delta function $\delta(\omega)$ to be the FT of the function that has the value $\frac{1}{2\pi}$ for all x, then the FT of the complex exponential function $\frac{1}{2\pi} e^{-i\omega_0 x}$ is $\delta(\omega - \omega_0)$, visualized as a "spike" at ω_0, that is, a generalized function that has the value $+\infty$ at $\omega = \omega_0$ and zero elsewhere. This is a useful result, in that it provides the motivation for considering the Fourier transform of a signal $s(t)$ containing hidden periodicities. If $s(t)$ is a sum of complex exponentials with frequencies $-\omega_n$ then its Fourier transform will consist of Dirac delta functions $\delta(\omega - \omega_n)$. If we then estimate the Fourier transform of $s(t)$ from sampled data, we are looking for the peaks in the Fourier transform that approximate the infinitely high spikes of these delta functions.

Exercise 6: Use the fact that $\text{sgn}(x) = 2u(x) - 1$ and the previous exercise to show that $f(x) = u(x)$ has the FT $F(\omega) = i/\omega + \pi\delta(\omega)$.

Generally, the functions $f(x)$ and $F(\omega)$ are complex-valued, so that we may speak about their real and imaginary parts. The next exercise explores the connections that hold among these real-valued functions.

Exercise 7: Let $f(x)$ be arbitrary and $F(\omega)$ its Fourier transform. Let $F(\omega) = R(\omega) + iX(\omega)$, where R and X are real-valued functions, and similarly, let $f(x) = f_1(x) + if_2(x)$, where f_1 and f_2 are real-valued. Find relationships between the pairs R,X and f_1,f_2.

Exercise 8: Let f, F be a FT pair. Let $g(x) = \int_{-\infty}^{x} f(y)dy$. Show that the FT of $g(x)$ is $G(\omega) = \pi F(0)\delta(\omega) + \frac{iF(\omega)}{\omega}$.

Hint: For $u(x)$ the Heaviside function we have

$$\int_{-\infty}^{x} f(y)dy = \int_{-\infty}^{\infty} f(y)u(x-y)dy.$$

We can use properties of the Dirac delta functions to extend the Parseval equation to Fourier transforms, where it is usually called the *Parseval-Plancherel* equation.

Exercise 9: Let $f(x), F(\omega)$ and $g(x), G(\omega)$ be Fourier transform pairs. Use equation (6.1) to establish the Parseval-Plancherel equation

$$\langle f, g\rangle = \int f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int F(\omega)\overline{G(\omega)}d\omega,$$

from which it follows that

$$||f||^2 = \langle f, f\rangle = \int |f(x)|^2 dx = \frac{1}{2\pi}\int |F(\omega)|^2 d\omega.$$

Exercise 10: We define the *even part* of $f(x)$ to be the function

$$f_e(x) = \frac{f(x) + f(-x)}{2},$$

and the *odd part* of $f(x)$ to be

$$f_o(x) = \frac{f(x) - f(-x)}{2};$$

define F_e and F_o similarly for F the FT of f. Let $F(\omega) = R(\omega) + iX(\omega)$ be the decomposition of F into its real and imaginary parts. We say that f is a *causal function* if $f(x) = 0$ for all $x < 0$. Show that, if f is causal, then R and X are related; specifically, show that X is the *Hilbert transform* of R, that is,

$$X(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R(\alpha)}{\omega - \alpha}d\alpha.$$

Hint: If $f(x) = 0$ for $x < 0$ then $f(x)\mathrm{sgn}(x) = f(x)$. Apply the convolution theorem, then compare real and imaginary parts.

Exercise 11: The one-sided *Laplace transform* (LT) of f is $\mathcal{F}$ given by

$$\mathcal{F}(z) = \int_{0}^{\infty} f(x)e^{-zx}dx.$$

Compute $\mathcal{F}(z)$ for $f(x) = u(x)$, the Heaviside function. Compare $\mathcal{F}(-i\omega)$ with the FT of u.

Chapter 7

The Fast Fourier Transform

A fundamental problem in signal processing is to estimate finitely many values of the function $F(\omega)$ from finitely many values of its (inverse) Fourier transform, $f(t)$. As we have seen, the DFT arises in several ways in that estimation effort. The *fast Fourier transform* (FFT), discovered in 1965 by Cooley and Tukey, is an important and efficient algorithm for calculating the vector DFT [74]. John Tukey has been quoted as saying that his main contribution to this discovery was the firm and often voiced belief that such an algorithm must exist.

To illustrate the main idea behind the FFT consider the problem of evaluating a real polynomial $P(x)$ at a point, say $x = c$: let the polynomial be

$$P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_{2K} x^{2K},$$

where a_{2K} might be zero. Performing the evaluation efficiently by Horner's method,

$$P(c) = (((a_{2K} c + a_{2K-1})c + a_{2K-2})c + a_{2K-3})c + ...,$$

requires $2K$ multiplications, so the complexity is on the order of the degree of the polynomial being evaluated. But suppose we also want $P(-c)$. We can write

$$P(x) = (a_0 + a_2 x^2 + ... + a_{2K} x^{2K}) + x(a_1 + a_3 x^2 + ... + a_{2K-1} x^{2K-2})$$

or

$$P(x) = Q(x^2) + xR(x^2).$$

Therefore we have $P(c) = Q(c^2) + cR(c^2)$ and $P(-c) = Q(c^2) - cR(c^2)$. If we evaluate $P(c)$ by evaluating $Q(c^2)$ and $R(c^2)$ separately, one more multiplication gives us $P(-c)$ as well. The FFT is based on repeated use of this idea, which turns out to be more powerful when we are using complex exponentials, because of their periodicity.

Say the data are the samples are $\{f(n\Delta), n = 1, ..., N\}$, where $\Delta > 0$ is the sampling increment or sampling spacing.

The DFT estimate of $F(\omega)$ is the function $F_{DFT}(\omega)$, defined for ω in $[-\pi/\Delta, \pi/\Delta]$, and given by

$$F_{DFT}(\omega) = \Delta \sum_{n=1}^{N} f(n\Delta) e^{in\Delta\omega}.$$

The DFT estimate $F_{DFT}(\omega)$ is data consistent; its inverse Fourier transform value at $t = n\Delta$ is $f(n\Delta)$ for $n = 1, ..., N$. The DFT is sometimes used in a slightly more general context in which the coefficients are not necessarily viewed as samples of a function $f(t)$.

Given the complex N-dimensional column vector $\mathbf{f} = (f_0, f_1, ..., f_{N-1})^T$ define the DFT of vector $\mathbf{f}$ to be the function $DFT_{\mathbf{f}}(\omega)$, defined for ω in $[0, 2\pi)$, given by

$$DFT_{\mathbf{f}}(\omega) = \sum_{n=0}^{N-1} f_n e^{in\omega}.$$

Let $\mathbf{F}$ be the complex N-dimensional vector $\mathbf{F} = (F_0, F_1, ..., F_{N-1})^T$, where $F_k = DFT_{\mathbf{f}}(2\pi k/N), k = 0, 1, ..., N-1$. So the vector $\mathbf{F}$ consists of N values of the function $DFT_{\mathbf{f}}$, taken at N equispaced points $2\pi/N$ apart in $[0, 2\pi)$.

From the formula for $DFT_{\mathbf{f}}$ we have, for $k = 0, 1, ..., N-1$,

$$F_k = F(2\pi k/N) = \sum_{n=0}^{N-1} f_n e^{2\pi i nk/N}. \tag{7.1}$$

To calculate a single F_k requires N multiplications; it would seem that to calculate all N of them would require N^2 multiplications. However, using the FFT algorithm we can calculate vector $\mathbf{F}$ in approximately $N \log_2(N)$ multiplications.

Suppose that $N = 2M$ is even. We can rewrite equation(7.1) as follows:

$$F_k = \sum_{m=0}^{M-1} f_{2m} e^{2\pi i(2m)k/N} + \sum_{m=0}^{M-1} f_{2m+1} e^{2\pi i(2m+1)k/N},$$

or, equivalently,

$$F_k = \sum_{m=0}^{M-1} f_{2m} e^{2\pi imk/M} + e^{2\pi ik/N} \sum_{m=0}^{M-1} f_{2m+1} e^{2\pi imk/M}. \tag{7.2}$$

Note that if $0 \leq k \leq M-1$ then

$$F_{k+M} = \sum_{m=0}^{M-1} f_{2m} e^{2\pi imk/M} - e^{2\pi ik/N} \sum_{m=0}^{M-1} f_{2m+1} e^{2\pi imk/M}, \tag{7.3}$$

so there is no additional computational cost in calculating the second half of the entries of $\mathbf{F}$, once we have calculated the first half. The FFT is the algorithm that results when take full advantage of the savings obtainable by splitting a DFT calculating into two similar calculations of half the size.

We assume now that $N = 2^L$. Notice that if we use equations (7.2) and (7.3) to calculate vector $\mathbf{F}$, the problem reduces to the calculation of two similar DFT evaluations, both involving half as many entries, followed by one multiplication for each of the k between 0 and $M-1$. We can split these in half as well. The FFT algorithm involves repeated splitting of the calculations of DFTs at each step into two similar DFTs, but with half the number of entries, followed by as many multiplications as there are entries in either one of these smaller DFTs. We use recursion to calculate the cost $C(N)$ of computing $\mathbf{F}$ using this FFT method. From equation (7.2) we see that $C(N) = 2C(N/2) + (N/2)$. Applying the same reasoning to get $C(N/2) = 2C(N/4) + (N/4)$, we obtain

$$\begin{aligned} C(N) &= 2C(N/2) + (N/2) = 4C(N/4) + 2(N/2) = ... \\ &= 2^L C(N/2^L) + L(N/2) = N + L(N/2). \end{aligned}$$

Therefore the cost required to calculate $\mathbf{F}$ is approximately $N \log_2 N$.

From our earlier discussion of discrete linear filters and convolution we see that the FFT can be used to calculate the periodic convolution (or even the non-periodic convolution) of finite length vectors.

Finally, let's return to the original context of estimating the Fourier transform $F(\omega)$ of function $f(t)$ from finitely many samples of $f(t)$. If we have N equispaced samples we can use them to form the vector $\mathbf{f}$ as above and perform the FFT algorithm to get vector $\mathbf{F}$ consisting of N values of the DFT estimate of $F(\omega)$. It may happen that we wish to calculate more than N values of the DFT estimate, perhaps to produce a smooth looking graph. We can still use the FFT, but we must trick it into thinking we have more data that the N samples we really have. We do this by *zero-padding.* Instead of creating the N-dimensional vector $\mathbf{f}$, we make a longer vector by appending, say, J zeros to the data, to make a vector that has dimension $N + J$. The DFT estimate is still the same function of ω, since we have only included new zero coefficients as fake data. But the FFT thinks we have $N + J$ data values, so it returns $N + J$ values of the DFT, at $N + J$ equispaced values of ω in $[0, 2\pi)$.

Chapter 8

Bandlimited Extrapolation

Let $f(x)$ and $F(\omega)$ be a Fourier transform pair. We know from the formulas in equations (4.1) and (4.2) that we can determine F from f and vice versa. But what happens if we have some, but not all, of the values $f(x)$? Can we still find $F(\omega)$ for all ω? If we can, then we can also recover the missing values of f, which says that there must be considerable redundancy in the way f stores information. We shall investigate this matter further now for the important case in which F has bounded support; that is, there is some $\Omega > 0$ such that $F(\omega) = 0$, for $|\omega| > \Omega$. The function $f(x)$ is then said to be Ω-*bandlimited*.

We shall assume throughout this chapter that f is Ω-bandlimited and ask how much we need to know about f to recover $F(\omega)$ for all ω. Because recovering $F(\omega)$ for all ω is equivalent to finding $f(x)$ for all x, this problem is called the *bandlimited extrapolation problem*.

We have already encountered one result along these lines. According to Shannon's sampling theorem, if we have the values $\{f(n\Delta), -\infty < n < \infty\}$, for some $\Delta \in (0, \frac{\pi}{\Omega}]$, then we can recover $F(\omega)$ for all ω and thereby $f(x)$ for all x. Therefore, these infinite sequences of samples of f contain complete information about f. Other results of this sort have quite a different flavor.

Since $F(\omega) = 0$ outside its interval of support $[-\Omega, \Omega]$ the extension of $f(x)$ to complex z, given by the *Fourier-Laplace transform*

$$f(z) = \int_{-\infty}^{\infty} F(\omega) e^{-iz\omega} d\omega / 2\pi, \tag{8.1}$$

can be differentiated under the integral sign since the limits of integration are now finite. In fact, the function $f(z)$ is a complex-valued function that is analytic throughout the complex plane. Such functions have power series expansions that converge for all z.

Exercise 1: Show that there can be no Fourier transform pair f, F for which positive constants a and b exist such that $f(x) = 0$ for $|x| > a$ and $F(\omega) = 0$ for $|\omega| > b$. Thus it is not possible for both f and F to be band-limited.

Hint: Use the analyticity of the function $f(z)$.

The coefficients needed for such a power series expansion are determined by the derivatives of $f(z)$ at a single point, say $z = 0$. Therefore, if we have the values of $f(z)$ for z in some small disc around $z = 0$ we have all the information we need. Actually, even this amount of knowledge about f is too much; to calculate the derivatives at $z = 0$ we need only know $f(x_n)$ for some sequence $\{x_n\}$ of real numbers converging to $z = 0$.

This is fine in theory, but, of course, we cannot hope to calculate all the derivatives of f at $z = 0$. Even calculating a few derivatives in the presence of noisy measurements of f is hopeless. In [152] Papoulis presents an iterative scheme for determining $F(\omega)$ from knowledge of $f(x)$ for x within an interval $A = [a, b]$ of the real line. This is not a practical technique, since it uses infinitely many samples of $f(x)$, but can be modified to provide useful algorithms, as we shall see. The iterative and non-iterative methods we describe below are usually called *super-resolution techniques* in the signal processing literature. Similar methods applied in sonar and radar array processing are called *super-directive* methods [75].

Papoulis' iterative method: Let $g^0(x) = \chi_A(x)f(x)$. Having found $g^k(x)$ let $G^k(\omega)$ be the FT of g^k, $H^k(\omega) = \chi_\Omega(\omega)G^k(\omega)$ and $h^k(x)$ the inverse FT of $H^k(\omega)$. Then take $g^{k+1}(x) = f(x)$ for $x \in A$ and $g^{k+1}(x) = h^k(x)$ otherwise. The sequence $\{h^k(x)\}$ converges to $f(x)$ for all x and the sequence $\{H^k\}$ converges in the mean square sense to F.

In practice we have only finitely many values of $f(x)$. This is not, of course, enough information to determine $F(\omega)$. We seek an estimate of F, or, equivalently, an approximate extrapolation of the data. We consider now several practical variants of Papoulis' iterative method.

Gerchberg-Papoulis iteration (I): The algorithm discussed in this section is called the *Gerchberg-Papoulis* (GP) bandlimited iteration method [100], [151]. For notational convenience we shall assume that $\Omega < \pi$ and that we have the finite data $f(n)$, $n = 0, 1, ..., M - 1$. We seek to estimate the values $f(n)$, $n = M, M + 1, ..., N$ for some choice of $N > M$. We begin with g^0 the N-dimensional vector with entries $g^0(n) = f(n)$ for

$n = 0, 1, ..., M-1$ and $g^0(n) = 0$ for $n = M, M+1, ..., N-1$. Then having found the vector g^k we let

$$G_m^k = \sum_{n=0}^{N-1} g^k(n) \exp(2\pi imn/N),$$

for $m = 0, 1, ..., N-1$. We interpret these values as samples of a function $G^k(\omega)$ defined on $[-\pi, \pi]$; specifically, we take

$$G_m^k = G^k(2\pi m/N)$$

for $m = 0, 1, ..., \frac{N}{2}$ and

$$G_m^k = G^k(-2\pi + 2\pi m/N)$$

for $m = \frac{N}{2} + 1, ..., N-1$; for convenience we assume that N is even. Mimicking the definition of $H^k(\omega)$, we define H_m^k to be G_m^k for those $m = 0, 1, ..., \frac{N}{2}$ such that $2\pi m/N \leq \Omega$ and for those $m = \frac{N}{2} + 1, ..., N-1$ for which $-2\pi + 2\pi m/N \geq -\Omega$. For all other values of m we set $H_m^k = 0$. Now calculate

$$h_n^k = \frac{1}{N} \sum_{m=0}^{N-1} H_m^k \exp(-2\pi imn/N),$$

for $n = 0, 1, ..., N-1$. Finally, set $g_n^{k+1} = f(n)$, for $n = 0, 1, ..., M-1$ and $g_n^{k+1} = h_n^k$ for $n = M, M+1, ..., N-1$. The limit vector g^∞ has $g_n^\infty = f(n)$ for $n = 0, 1, ..., M-1$, but in order to have $G_m^\infty = 0$ for those m corresponding to frequencies outside $[-\Omega, \Omega]$ we need to take $N \geq M\pi/\Omega$. The values g_n^∞ for $n = M, M+1, ..., N-1$ are then our extrapolated values of f.

The advantages of this approach are that only finite data is used and the calculations can be performed using the fast Fourier transform. The vectors obtained are optimal in some sense [53], [54]. Obviously, one drawback is that we do not extrapolate $f(n)$ for all integers n, but only for a finite subset. Also, we do not obtain a function $G^\infty(\omega)$ of the continuous variable ω that is equal to zero for all ω outside the band $[-\Omega, \Omega]$ and whose corresponding $g^\infty(x)$ is consistent with the finite data. To remedy this we consider another variant of the GP algorithm.

Gerchberg-Papoulis iteration (II): We shall assume again that $\Omega < \pi$ and that we have the finite data $f(n)$, $n = 0, 1, ..., M-1$. Since

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) \exp(in\omega)$$

for $\omega \in [-\pi, \pi]$, we seek to extrapolate $f(n)$ for n not in the set $\{0, 1, ..., M-1\}$.

Mimicking the algorithm in the previous section, we begin with the infinite sequence $g^0 = \{g_n^0, -\infty < n < \infty\}$ where $g_n^0 = f(n)$ for $n = 0, 1, ..., M-1$ and $g_n^0 = 0$ otherwise. Having found the infinite sequence g^k we define

$$G^k(\omega) = \sum_{n=-\infty}^{\infty} g_n^k \exp(in\omega)$$

for $\omega \in [-\pi, \pi]$. Then we set

$$H^k(\omega) = \chi_\Omega(\omega) G^k(\omega)$$

and

$$h_n^k = \frac{1}{2\pi} \int_{-\pi}^{\pi} H^k(\omega) \exp(-in\omega) d\omega.$$

Then let $g_n^{k+1} = f(n)$ for $n = 0, 1, ..., M-1$ and $g_n^{k+1} = h_n^k$ otherwise.

It would appear that this iterative scheme cannot actually be performed because it requires calculating g_n^{k+1} for all integers n. Fortunately, there is a way out.

Non-iterative bandlimited extrapolation: Note that $G^{k+1}(\omega)$ can be written as

$$G^{k+1}(\omega) = H^k(\omega) + G^0(\omega) - \sum_{n=0}^{N-1} h_n^k \exp(in\omega),$$

so that

$$H^{k+1}(\omega) - H^k(\omega) = \chi_\Omega(\omega) \sum_{n=0}^{N-1} a_n^k \exp(in\omega) \tag{8.2}$$

for some $a_0^k, ..., a_{N-1}^k$. If we wish we can implement the GP iterative method by iteratively updating these constants. There is a better way to proceed, however.

It follows from equation (8.2) and the definition of H^0 that the limit $H^\infty(\omega)$ has the form

$$H^\infty(\omega) = \chi_\Omega(\omega) \sum_{n=0}^{N-1} a_n \exp(in\omega) \tag{8.3}$$

for some constants $a_0, ..., a_{N-1}$. We then solve for these coefficients using our data. Taking the inverse Fourier transform of both sides of equation (8.3)and forcing data consistency, we obtain the system of equations

$$f(m) = \sum_{n=0}^{N-1} a_n \frac{\sin \Omega(m-n)}{\pi(m-n)}, \tag{8.4}$$

$m = 0, ..., N-1$, which we solve to find the coefficients. Once we have the coefficients we insert them into the expression for $H^{\infty}(\omega)$ to obtain a function supported on the interval $[-\Omega, \Omega]$ whose associated $h^{\infty}(x)$ is consistent with the data. The extrapolated sequence is then $\{h^{\infty}(n)\}$ for integers n not between 0 and $M-1$. This noniterative implementation of the GP extrapolation is not new; it was presented in [45], and has been rediscovered several times since then (see p. 209 of [170]).

Because our data usually contains noise we need to exercise some care in solving the system in equation (8.4). The matrix S whose entries are

$$S_{mn} = \frac{\sin \Omega(m-n)}{\pi(m-n)}$$

is typically ill-conditioned, particularly when Ω is much smaller than π. To reduce sensitivity to noise we can *regularize*; one way is to multiply the entries on the main diagonal of S by, say, 1.0001. This increases the eigenvalues of S, thereby decreasing the eigenvalues of S^{-1} and making the computed solution less sensitive to the noise.

The finite data we have tells us nothing about the values $f(n)$ we have not measured, in the sense that we can define $f(M)$ any way we wish and still construct an Ω-bandlimited function consistent with the data and with this chosen value of $f(M)$. In a similar sense our finite data also tells us nothing about the value of Ω; we can select any interval $[a, b]$ and find a function $H(\omega)$ supported on $[a, b]$ whose $h(x)$ is consistent with the data. But this is not quite the whole story; finite data cannot rule out anything, but it can suggest strongly that certain things are false. For example, if we select the interval $[a, b]$ disjoint from $[-\Omega, \Omega]$ the function $H(\omega)$ will probably have large energy; that is, the integral

$$\int_a^b |H(\omega)|^2 d\omega$$

will be much larger than

$$\int_{-\Omega}^{\Omega} |H^{\infty}(\omega)|^2 d\omega.$$

We can use this fact to help us decide if we have chosen a good value for Ω. In [43] this same idea was used to obtain an iterative algorithm for solving the phase retrieval problem discussed in a later chapter.

When the data set is large, as usually happens in multi-dimensional problems such as image reconstruction, solving the equations (8.4) is sometimes performed iteratively. Nevertheless, the algorithm still differs from the first GP method in that we are still extrapolating infinitely many values of $f(n)$; we are just doing it using a finite parameter model.

The non-iterative implementation of the Gerchberg-Papoulis bandlimited extrapolation method can be extended in several ways to solve Fourier transform estimation problems. The *modified* DFT (MDFT) estimator generalizes the non-iterative GP method to accomodate non-equispaced sampling. More generally, the PDFT method permits us to include other prior information about the shape of $F(\omega)$ beyond knowledge of its support; it also applies to multi-dimensional problems. Constructing the matrix used in the system of equations can be difficult when the data sets are large; an iterative discrete implementation of the PDFT, the DPDFT, allows us to avoid dealing with this large matrix. There is also a nonlinear version of the PDFT, the *indirect* PDFT (IPDFT), that extends the maximum entropy method for extrapolating autocorrelation data.

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