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An Intuitive Derivation of Heron's Formula

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From elementary geometry we learn that two triangles are congruent if their edges have the same lengths, so it should come as no surprise that the edge lengths of a triangle determine the *area* of that triangle. On the other hand, the explicit formula for the area of a triangle in terms of its edge lengths, named for Heron of Alexandria (although attributed to Archimedes [**4**]), seems to be less commonly remembered (as compared with, say, the formulas for the volume of a sphere or the area of a rectangle).

One reason why Heron's formula is so easily forgotten may be that proofs are usually presented as unwieldy verifications of an already known formula, rather than as expositions that derive a formula from scratch in a constructive and intuitive manner. Perhaps the derivation that follows, while not truly elementary, will render Heron's formula more memorable for its symmetric and intuitive factorization.

The first step of this derivation is to recall that the square of the area of a triangle is a *polynomial* in the edge lengths. More generally, suppose that T is a simplex in $\mathbb{R}^n$ with vertices $x_0, x_1, \ldots, x_n$, where $x_0 = 0$, the origin. Let A denote the $n \times n$ matrix whose columns are given by the vectors $x_1, \ldots, x_n$, and suppose that the x_i are ordered so that A has positive determinant. The volume of T is then given by $\det(A) = n!\, V(T)$, implying that

$$V(T)^2 = \frac{1}{(n!)^2} \det(A^t A), \tag{1}$$

where A^t is the transpose of the matrix A. The entries of the matrix $A^t A$ are dot products of the form $x_i \cdot x_j$. From the identity

$$x_i \cdot x_j = \frac{1}{2}\left(|x_i|^2 + |x_j|^2 - |x_i - x_j|^2\right) \tag{2}$$

it then follows that the value of $V(T)^2$ is a *polynomial* in the *squares* of the edge lengths of T. Said differently, if T has edge lengths $a_{ij} = |x_i - x_j|$, then $V(T)^2$ is a polynomial in the variables $b_{ij} = a_{ij}^2$, as well as in the variables a_{ij} themselves. Since the determinant of an $n \times n$ matrix is a homogeneous polynomial of degree n in the matrix entries, the polynomial $f(a_{ij}) = V(T)^2$ is a *homogeneous polynomial of degree* $2n$. This polynomial is sometimes formulated in terms of linear algebraic expressions such as Cayley-Menger determinants [**2**]. In certain instances, however, the polynomial f also admits factorization into linear or quadratic irreducible factors.

Heron's formula concerns the two-dimensional case, a formula for the *area* $A(T)$ of a triangle T. In this case $A(T)^2 = f(a, b, c)$, a homogeneous polynomial of degree four in the edge lengths a, b, and c having real coefficients. But how, if at all, can this polynomial f be factored?

A key observation in dimension two is that the area of a triangle is invariant under rotations and reflections: the order of the edges a, b, and c does not matter when computing the area. More specifically, we orient the triangle T so that the edge of length a is parallel to the x-axis, while a, b, and c label the edge lengths of T in counterclockwise order. If we reflect T across the y-axis, the labels b and c are exchanged, whereas the squared area $A(T)^2 = f(a, b, c)$ remains the same. Since this holds for *all* values of a, b, and c, the polynomial $f(a, b, c)$ must be *symmetric* in the parameters b and c. A similar argument using rotation implies that the polynomial $f(a, b, c)$ is symmetric in all three variables a, b, and c, a property that is not at all obvious from (1) and (2). (Indeed, this is no longer true for the volume of a general n-simplex if $n \geq 3$.)

To find a factorization of f into irreducible polynomials, we attempt to discover its roots; that is, to determine where the polynomial vanishes. For the polynomial f this is determined in part by the triangle inequality. Specifically, if a triangle has edge lengths a, b, and c, then $a + b \geq c$. Since equality occurs when the triangle flattens to a line segment (that is, to a degenerate triangle *having area zero*), the linear polynomial $a + b - c$ is suggested as a possible factor of f.

To verify that $a + b - c$ is indeed a factor, we view f as a polynomial in the ring $\mathbb{R}[b, c][a]$ (i.e., as a polynomial in the variable a having coefficients in $\mathbb{R}[b, c]$). Division with remainder then yields

$$f(a, b, c) = (a + b - c)g(a, b, c) + r(b, c), \tag{3}$$

for some g in $\mathbb{R}[a, b, c]$ and r in $\mathbb{R}[b, c]$.

Note that a does not appear in the polynomial expression for r. Assume that $c \geq b \geq 0$, and set $a = c - b$. In this instance the values of a, b, and c are the edge lengths of a degenerate triangle, so that $f(a, b, c) = A(T)^2 = 0$. It follows from (3) that $r(b, c) = 0$ whenever $c \geq b \geq 0$. In other words, $r(b, c)$ vanishes on a subset of $\mathbb{R}^2$ having nonempty interior. Since r is a polynomial, it follows that r is identically zero, meaning that

$$f(a, b, c) = (a + b - c)g(a, b, c).$$

Thus $a + b - c$ divides f in $\mathbb{R}[a, b, c]$.

By a similar (and symmetric) argument, each of the linear polynomials $a - b + c$ and $-a + b + c$ also divides f in $\mathbb{R}[a, b, c]$. These linear polynomial factors of f are each irreducible, hence pairwise relatively prime. It follows from unique factorization in $\mathbb{R}[a, b, c]$ that

$$f(a, b, c) = (a + b - c)(a - b + c)(-a + b + c)h(a, b, c),$$

for some h in $\mathbb{R}[a, b, c]$.

Since the *symmetric* polynomial f is homogeneous of degree four, the polynomial h must be symmetric and homogeneous of degree one, so $h = (a + b + c)k$ for some real constant k. Accordingly,

$$f(a, b, c) = (a + b - c)(a - b + c)(-a + b + c)(a + b + c)k,$$

where the constant k is independent of the triangle T. Evaluating $f(3,4,5) = A(T)^2 = 36$ for the easy case of a 3-4-5 right triangle implies that $k = 1/16$, so that

$$f(a,b,c) = \frac{1}{16}(a+b-c)(a-b+c)(-a+b+c)(a+b+c). \tag{4}$$

It is customary to define the *semiperimeter* $s(T)$ of a triangle T by $s(T) = (a+b+c)/2$ (that is, $s(T)$ is half the perimeter of T). Equation (4) then yields the following classical result [**4**]:

Theorem 1 (Heron's Formula). *If a triangle T has edge lengths a, b, and c and semiperimeter s, then*

$$A(T)^2 = s(s-a)(s-b)(s-c).$$

More generally, it follows from (1) and (2) that the square $V(T)^2$ of the volume of an n-dimensional simplex T, where $n \geq 3$, is also polynomial in the squares of the edge lengths of T. Unfortunately $V(T)^2$ is *not* typically symmetric in all of its variables, since some pairs of edges of T are incident, while others are not. However, there do exist nontrivial factorizations of $V(T)^2$ when a simplex T satisfies certain geometric symmetry conditions.

A three-dimensional tetrahedron T is said to be *isosceles* (or *disphenoid* [**1**, p. 15]) if its four triangular facets are all congruent to one another (say, all congruent to a triangle having edge lengths a, b, and c). Equivalently, one may define a tetrahedron T to be isosceles if its three pairs of opposing (nonincident) edges in T have common lengths, call them a, b, and c (see Figure 1).

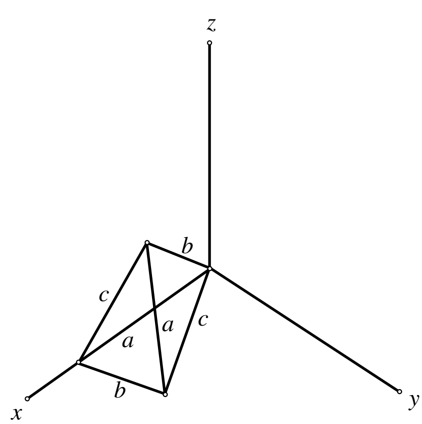

Figure 1. An isosceles tetrahedron.

Theorem 2 (Volume of an Isosceles Tetrahedron). *If T is an isosceles tetrahedron having congruent opposite edge pairs of length a, b, and c, then*

$$V(T)^2 = \frac{1}{72}(a^2+b^2-c^2)(a^2-b^2+c^2)(-a^2+b^2+c^2). \tag{5}$$

A synthetic proof of (5) can be found in [**3**, p. 101]. Instead we give an algebraic proof, following the same technique as for Heron's formula (Theorem 1).

Proof of Theorem 2. The squared volume $V(T)^2$ is a homogeneous polynomial in the squares of the lengths of the edges of T. Since T is isosceles, this implies that

$$V(T)^2 = f(a^2, b^2, c^2).$$

The symmetries of an isosceles tetrahedron once again imply that f is a *symmetric* polynomial. To see this, position T pointing upward with one facet in the xy-plane, with an edge labelled a parallel to the x-axis, and edges a, b, and c of the bottom facet labelled counterclockwise when viewed from above, as in Figure 1. Once again a reflection of T will exchange the roles of b and c, while preserving the volume $V(T)$ and the value of f. Similarly, volume-preserving rotations and reflections will permute the roles of a, b, and c in every possible way, without affecting the value of the polynomial f. Finally, recall that volume in $\mathbb{R}^3$ is homogeneous of degree three with respect to edge lengths, so the function f is also a homogeneous polynomial of degree three in the variables a^2, b^2, and c^2.

The factors of f can now be determined by considering the cases in which the volume of T is zero, namely, when the tetrahedron T is flat or otherwise degenerate. When T is isosceles, this can occur only if T is a *rectangle*. If the triangular facets of T are congruent to a triangle F having edge lengths satisfying $c \geq b \geq a$, then T is degenerate (having volume zero) if and only if T is a rectangle having side lengths a and b and diagonal length c. In this instance, we have $c^2 = a^2 + b^2$ by the Pythagorean theorem. This suggests that $a^2 + b^2 - c^2$ may be a factor of the polynomial f.

To verify this, denote $A = a^2$, $B = b^2$, and $C = c^2$. The previous observations imply that f is a homogeneous polynomial in the variables A, B, and C. Division with remainder then yields $f(A, B, C) = (A + B - C)g(A, B, C) + r(B, C)$ for some g in $\mathbb{R}[A, B, C]$ and r in $\mathbb{R}[B, C]$.

The variable A does not appear in the polynomial expression for r. Moreover, it follows from our earlier observations that, if $A + B - C = 0$, then $f(A, B, C) = V^2 = 0$. Hence, $r(B, C) = 0$ whenever $C \geq B \geq 0$ (simply set $A = C - B$). Since r is a polynomial, it follows that r is identically zero, which gives

$$f(A, B, C) = (A + B - C)g(A, B, C).$$

In other words, $A + B - C$ divides f in $\mathbb{R}[A, B, C]$. Symmetry implies that $A - B + C$ and $-A + B + C$ also divide f, so that

$$f(A, B, C) = (A + B - C)(A - B + C)(-A + B + C)h(A, B, C) \tag{6}$$

for some h in $\mathbb{R}[A, B, C]$.

Since f is a homogeneous polynomial of degree three in the variables A, B, and C, it follows that h is a constant. Recall that the volume of the regular tetrahedron of unit edge length $a = b = c = 1$ is $\sqrt{2}/12$. It follows that $1/72 = f(1, 1, 1) = h$. Resubstituting $A = a^2$, $B = b^2$, and $C = c^2$ in (6) yields (5). ■

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