Containment and inscribed simplices

Daniel A. Klain
Department of Mathematical Sciences
University of Massachusetts Lowell
Lowell, MA 01854 USA
Daniel_Klain@uml.edu

Abstract Let K and L be compact convex sets in $\mathbb{R}^n$. The following two statements are shown to be equivalent:

(i) For every polytope $Q \subseteq K$ having at most $n+1$ vertices, L contains a translate of Q.

(ii) L contains a translate of K.

Let $1 \leq d \leq n-1$. It is also shown that the following two statements are equivalent:

(i) For every polytope $Q \subseteq K$ having at most $d+1$ vertices, L contains a translate of Q.

(ii) For every d-dimensional subspace ξ, the orthogonal projection L_ξ of the set L contains a translate of the corresponding projection K_ξ of the set K.

It is then shown that, if K is a compact convex set in $\mathbb{R}^n$ having at least $d+2$ exposed points, then there exists a compact convex set L such that every d-dimensional orthogonal projection L_ξ contains a translate of the projection K_ξ, while L does not contain a translate of K. In particular, if $\dim K > d$, then there exists L such that every d-dimensional projection L_ξ contains a translate of the projection K_ξ, while L does not contain a translate of K.

This note addresses questions related to following general problem: Consider two compact convex subsets K and L of n-dimensional Euclidean space. Suppose that, for a given dimension $1 \leq d < n$, every d-dimensional orthogonal projection (shadow) of L contains a translate of the corresponding projection of K. Under what conditions does it follow that the original set L contains a translate of K? In other words, if K can be translated to "hide behind" L from any perspective, does it follow that K can "hide inside" L?

This question is easily answered when a sufficient degree of symmetry is imposed. For example, a support function argument implies that the answer is *Yes* if *both* of the bodies K and L are centrally symmetric. It is also not difficult to show that if every d-projection of K (for some $1 \leq d < n$) can be translated into the corresponding shadow of an orthogonal n-dimensional box C, then K fits inside C by some translation, since one needs only to check that the widths are compatible in the n edge directions of C. A similar observation applies if

C is a parallelotope (an affine image of a box), a cylinder (the product of an $(n-1)$-dimensional compact convex set with a line segment), or a similarly decomposable product set; see also [8]).

For more general classes of convex bodies the situation is quite different. Given any $n > 1$ and $1 \leq d \leq n-1$, it is possible to find convex bodies K and L in $\mathbb{R}^n$ such every d-dimensional orthogonal projection (shadow) of L contains a translate of the corresponding projection of K, even though K has *greater volume* than L (and so certainly could not fit inside L). For a detailed example of this volume phenomenon, see [8].

In [11] Lutwak uses Helly's theorem to prove that, if every n-simplex containing L also contains a translate of K, then L contains a translate of K. In the present note we describe a dual result, by which the question of containment is related to properties of the *inscribed* simplices (and more general polytopes) of the bodies K and L. We then generalize these containment (covering) theorems in order to reduce questions about shadow (projection) covering to questions about inscribed simplices and related polytopes. Specifically we establish the following:

(1) Let K and L be compact convex sets in $\mathbb{R}^n$. The following are equivalent:
   (i) For every polytope $Q \subseteq K$ having at most $n+1$ vertices, L contains a translate of Q.
   (ii) L contains a translate of K.
   (Theorem 1.1)

(2) Let K and L be compact convex sets in $\mathbb{R}^n$, and let $1 \leq d \leq n-1$. The following are equivalent:
   (i) For every polytope $Q \subseteq K$ having at most $d+1$ vertices, L contains a translate of Q.
   (ii) For every d-dimensional subspace ξ, the orthogonal projection L_ξ contains a translate of K_ξ.
   (Theorem 1.3)

(3) Let $1 \leq d \leq n-1$. If K is a compact convex set in $\mathbb{R}^n$ having at least $d+2$ exposed points, then there exists a compact convex set L such that every d-dimensional orthogonal projection L_ξ contains a translate of the projection K_ξ, while L does not contain a translate of K itself. (Theorem 2.7)

In particular, if $\dim K > d$, then there exists L such that every d-shadow L_ξ contains a translate of the shadow K_ξ, while L does not contain a translate of K.

In this note we address the existence of a compact convex set L, whose shadows can cover those of a given set K, without containing a translate of K itself. A reverse question is addressed in [9]: Given a body L, does there necessarily exist K so that the shadows of L can cover those of K, while L does not contain a translate of K? These containment and covering problems are special cases of the following more general question: Under what conditions will a compact convex set necessarily contain a translate or otherwise congruent copy of another? Progress on different aspects of this general question also appears in the work of Gardner and Volčič [3], Groemer [4], Hadwiger [5, 6, 7, 10, 13], Jung [1, 16], Lutwak [11], Rogers [12], Soltan [15], Steinhagen [1, p. 86], Zhou [17, 18], and many others (see also [2, 8, 9]).

0. Background

Denote n-dimensional Euclidean space by $\mathbb{R}^n$, and let $\mathbb{S}^{n-1}$ denote the set of unit vectors in $\mathbb{R}^n$; that is, the unit $(n-1)$-sphere centered at the origin.

Let $\mathscr{K}_n$ denote the set of compact convex subsets of $\mathbb{R}^n$. If u is a unit vector in $\mathbb{R}^n$, denote by K_u the orthogonal projection of a set K onto the subspace $u^\perp$. More generally, if ξ is a d-dimensional subspace of $\mathbb{R}^n$, denote by K_ξ the orthogonal projection of a set K onto the subspace ξ. The boundary of a compact convex set K will be denoted by ∂K.

Let $h_K : \mathbb{R}^n \to \mathbb{R}$ denote the support function of a compact convex set K; that is,

$$h_K(v) = \max_{x \in K} x \cdot v$$

For $K, L \in \mathscr{K}_n$, we have $K \subseteq L$ if and only if $h_K \leq h_L$. If ξ is a subspace of $\mathbb{R}^n$ then the support function h_{K_ξ} is given by the restriction of h_K to ξ (see also [14, p. 38]).

If u is a unit vector in $\mathbb{R}^n$, denote by K^u the support set of K in the direction of u; that is,

$$K^u = \{x \in K \mid x \cdot u = h_K(u)\}.$$

If P is a convex polytope, then P^u is the face of P having u in its outer normal cone. A point $x \in \partial K$ is an *exposed point* of K if $x = K^u$ for some direction u. In this case, the direction u is said to be a *regular unit normal* to K. If K has non-empty interior, then the regular unit normals to K are dense in the unit sphere $\mathbb{S}^{n-1}$ (see [14, p. 77]).

Suppose that $\mathscr{F}$ is a family of compact convex sets in $\mathbb{R}^n$. Helly's Theorem [1, 10, 14, 16] asserts that, if every $n+1$ sets in $\mathscr{F}$ share a common point, then the entire family shares a common point. In [11] Lutwak used Helly's theorem to prove the following fundamental criterion for whether a set $L \in \mathscr{K}_n$ contains a translate of another compact convex set K.

Theorem 0.1 (Lutwak's Containment Theorem)**.** *Let $K, L \in \mathscr{K}^n$. The following are equivalent:*

(i) *For every simplex Δ such that $L \subseteq \Delta$, there exists $v \in \mathbb{R}^n$ such that $K + v \subseteq \Delta$.*
(ii) *There exists $v_0 \in \mathbb{R}^n$ such that $K + v_0 \subseteq L$.*

In other words, if every n-simplex containing L also contains a translate of K, then L contains a translate of K.

1. Inscribed polytopes and shadows

The following theorem provides an *inscribed* polytope counterpart to Lutwak's theorem.

Theorem 1.1 (Inscribed Polytope Containment Theorem)**.** *Let $K, L \in \mathscr{K}^n$. The following are equivalent:*

(i) *For every polytope $Q \subseteq K$ having at most $n+1$ vertices, there exists $v \in \mathbb{R}^n$ such that $Q + v \subseteq L$.*
(ii) *There exists $v_0 \in \mathbb{R}^n$ such that $K + v_0 \subseteq L$.*

Proof. The implication **(ii)** $\Rightarrow$ **(i)** is obvious. We show that **(i)** $\Rightarrow$ **(ii)**.

Note that $x + v \in L$ if and only if $v \in L - x$. If $x_0, x_1, \ldots, x_n \in K$, let Q denote the convex hull of these points. Note that Q has at most $n+1$ vertices. By the assumption **(i)** there exists v such that $Q + v \subseteq L$. In other words, $x_i + v \in L$ for each i, so that

$$v \in \bigcap_{i=0}^{n} (L - x_i). \tag{1}$$

Let $\mathscr{F} = \{L - x \mid x \in K\}$. By (1), $\mathscr{F}$ is a family of compact convex sets that satisfies the intersection condition of Helly's theorem [14, 16]. Hence there exists a point v_0 such that

$$v_0 \in \bigcap_{x \in K} (L - x).$$

In other words, $x + v_0 \in L$ for all $x \in K$, so that $K + v_0 \subseteq L$. □

Corollary 1.2. *Suppose that $K, L \in \mathscr{K}_n$ have* non-empty interiors. *If every* simplex *contained in K can be translated inside L, then K can be translated inside L.*

Proof. The proof is the same as that of Theorem 1.1, except that we must address the case in which the points $x_0, x_1, \ldots, x_n \in K$ are affinely dependent (and are not the vertices of a simplex).

In this case, since K has interior, perturbations of these points by a small distance $\epsilon > 0$ will yield the vertices of a simplex and a vector v_ϵ such that (1) holds for the perturbed points. As $\epsilon \to 0$ a vector v is obtained so that (1) holds for the original points $x_0, x_1, \ldots, x_n$ as well, since L is compact. Helly's theorem now applies, as in the previous proof. □

Theorem 1.1 is now generalized to address covering of lower-dimensional shadows.

Theorem 1.3 (Generalized Inscribed Polytope Containment Theorem)**.** *Let $K, L \in \mathscr{K}^n$, and suppose $1 \leq d \leq n$. The following are equivalent:*

- **(i)** *For every polytope $Q \subseteq K$ having at most $d+1$ vertices, there exists $v \in \mathbb{R}^n$ such that $Q + v \subseteq L$.*
- **(ii)** *For every d-dimensional subspace ξ, there exists $v \in \xi$ such that $K_\xi + v \subseteq L_\xi$.*

When K and L have non-empty interiors, this theorem can be reformulated in the following way: if every d-simplex contained in K can be translated into L, then every d-shadow of K can be translated into the corresponding d-shadow of L, and vice versa. In this case a perturbation argument applies, as in the proof of Corollary 1.2.

The next three lemmas will be used to prove Theorem 1.3.

Lemma 1.4. *Let T be an n-simplex, and let Q be a polytope in $\mathbb{R}^n$ having at most n vertices. Suppose that, for every unit vector u, there exists $v \in u^\perp$ such that $Q_u + v \subseteq T_u$. Then there exists $v_0 \in \mathbb{R}^n$ such that $Q + v_0 \subseteq T$.*

Proof. Since T has interior, ϵQ can be translated inside T for sufficiently small $\epsilon > 0$. Let $\hat{\epsilon}$ denote the maximum of all such $\epsilon > 0$. We will show that $\hat{\epsilon} \geq 1$, thereby proving the lemma.

Without loss of generality, translate T so that $\hat{\epsilon} Q \subseteq T$. If $\hat{\epsilon} Q$ does not intersect a given facet F of T, then some translate of $\hat{\epsilon} Q$ lies in the *interior* of T. This violates the maximality of $\hat{\epsilon}$. It follows that $\hat{\epsilon} Q$ must meet every facet of T. In particular, the vertex set of $\hat{\epsilon} Q$ must meet every facet of T. Since $\hat{\epsilon} Q$ has at most n vertices, while T has $n+1$ facets, some vertex of $\hat{\epsilon} Q$ must meet a face σ of T having co-dimension 2, where $\sigma = F_1 \cap F_2$, the intersection of two facets of T.

Let ℓ denote the line segment (i.e. the edge) complementary to σ in the boundary ∂T (so that T is the convex hull of the union $\ell \cup \sigma$). If $v \in \mathbb{R}^n$ points in the direction of ℓ, then T_v is an $(n-1)$-simplex. Moreover, every facet of T_v except one is exactly the projection of a facet of T, while $(F_1)_v = (F_2)_v = T_v$. The remaining facet of T_v is the projection σ_v of the ridge σ in T. Since $\hat{\epsilon} Q$ meets every facet of T, as well as the ridge σ, the projection $\hat{\epsilon} Q_v$ meets every facet of T_v, and is therefore inscribed (maximally) in T_v. Therefore, if $\epsilon > \hat{\epsilon}$, then ϵQ_v cannot be translated inside T_v. Since every shadow $Q_v = 1 Q_v$ of Q can be translated inside the corresponding shadow of T_v (by hypothesis), it follows that $\hat{\epsilon} \geq 1$. □

Lemma 1.5. *Let $L \in \mathscr{K}_n$, and let Q be a polytope in $\mathbb{R}^n$ having at most n vertices. If every shadow L_u contains a translate of the corresponding shadow Q_u, then L contains a translate of Q.*

Proof. Let T be an n-simplex that contains L. Since Q_u can be translated inside the corresponding shadow L_u, for each u, it follows that Q_u can be translated inside the corresponding shadow $T_u \supseteq L_u$ as well. By Lemma 1.4, Q can be translated inside T. Since this holds for every n-simplex $T \supseteq L$, Lutwak's Theorem 0.1 implies that L contains a translate of Q. □

Lemma 1.6. *Let $L \in \mathscr{K}_n$, and let Q be a polytope in $\mathbb{R}^n$ having at most $d + 1$ vertices, where $d < n$. Suppose that, for every d-dimensional subspace ξ, there exists $v \in \xi$ such that $Q_\xi + v \subseteq L_\xi$. Then there exists $v_0 \in \mathbb{R}^n$ such that $Q + v_0 \subseteq L$.*

Proof. Fix d and proceed by induction on n, starting with the case $n = d + 1$, which follows from Lemma 1.5.

Now suppose that Lemma 1.6 is true for $n \leq d + i$. If $n = d + i + 1$, then each projection Q_u also has at most $d + 1$ vertices, The induction assumption (in the lower dimensional space $u^\perp$) applies to Q_u, so that Q_u can be translated inside L_u for all u. Because Q has at most $d + 1 \leq n$ vertices, Lemma 1.5 implies that Q can be translated inside L. □

We now prove Theorem 1.3.

Proof of Theorem 1.3. To begin suppose that **(i)** holds. If $Q \subseteq K_\xi$ has at most $d + 1$ vertices, then Q is the projection of a polytope $\tilde{Q} \subseteq K$ having at most $d + 1$ vertices. By **(i)** there exists $v \in \mathbb{R}^n$ such that $\tilde{Q} + v \subseteq L$. By the linearity of orthogonal projection it follows that $Q + v_\xi \subseteq L_\xi$. The assertion **(ii)** now follows from Theorem 1.1 applied inside the subspace ξ.

To prove the converse, suppose that **(ii)** holds. Let $Q \subseteq K$ be a polytope with at most $d + 1$ vertices. For each ξ there exists $w \in \xi$ such that $K_\xi + w \subseteq L_\xi$, by **(ii)**. Since $Q \subseteq K$, we have $Q_\xi + w \subseteq K_\xi + w \subseteq L_\xi$ as well. It follows from Lemma 1.6 that there exists $v \in \mathbb{R}^n$ such that $Q + v \subseteq L$. □

Webster [16, p. 301] shows that if every triangle inside a compact convex set K can be translated inside a compact convex set L of the *same diameter* as K, then K can itself be translated inside L. Combining this observation with Theorem 1.3 yields the following corollary.

Corollary 1.7. *Let $K, L \in \mathscr{K}_n$, and let $d \geq 2$. Suppose that every d-dimensional shadow L_ξ contains a translate of the corresponding shadow K_ξ. If K and L have the same diameter, then L contains a translate of K.*

Webster's observation can be generalized in other ways via Theorem 1.3. Denote by $W(K)$ the *mean width* of the body K, taken over all directions in $\mathbb{R}^n$. If h_K is the support function of K, then

$$W(K) = \frac{2}{n\omega_n} \int_{\mathbb{S}^{n-1}} h_K(u)\, du,$$

where ω_n is the volume of the n-dimensional Euclidean unit ball. Evidently $W(K)$ is strictly monotonic, in the sense that $W(K) \leq W(L)$ whenever $K \subseteq L$, with equality if and only if $K = L$. (This follows from the fact that a compact convex set is uniquely determined by its support function [1, 14].) An alternative way to compute the mean width is given by the following Kubota-type formula [1, 10, 14]:

$$W(K) = \int_{G(n,2)} W(K_\xi)\, d\xi, \tag{2}$$

where $G(n,2)$ is the Grassmannian of 2-dimensional subspaces of $\mathbb{R}^n$, and the integral is taken with respect to Haar probability measure.

Corollary 1.8. *Let $K, L \in \mathscr{K}_n$. Suppose that every triangle inside K can be translated inside L. If K and L have the same mean width, then K and L are translates.*

Proof. By Theorem 1.3, every 2-dimensional shadow of K can be translated inside the corresponding shadow of L. It follows that $W(K_\xi) \leq W(L_\xi)$ for each 2-subspace ξ. If $W(K) = W(L)$, then (2) and the monotonicity of W yields

$$W(K) = \int_{G(n,2)} W(K_\xi)\, d\xi \leq \int_{G(n,2)} W(L_\xi)\, d\xi = W(L) = W(K),$$

so that equality $W(K_\xi) = W(L_\xi)$ holds in every 2-subspace ξ. The strictness of monotonicity for W now implies that each L_ξ is a *translate* of K_ξ.

A well-known theorem asserts that if K and L have translation-congruent 2-dimensional projections, then K and L are translates (see, for example, [2, p. 100] or [4, 7, 12]). □

The concept of mean width can be generalized to quermassintegrals (mean d-volumes of d-dimensional shadows). The previous argument (combining Theorem 1.3 with monotonicity, Kubota formulas, and the homothetic projection theorem) generalizes to give the following.

Corollary 1.9. *Let $K, L \in \mathscr{K}_n$, and let $d \geq 2$. Suppose that every d-simplex inside K can be translated inside L. If K has the same m-quermassintegral as L, for some $1 \leq m \leq d$, then K and L are translates.*

The previous corollary does not hold for $m > d$. For example, there exist convex bodies K and L in $\mathbb{R}^3$ such that L contains a translate of every triangle inside K, even though L has *strictly smaller* volume than K. Explicit examples of this phenomenon are described in [8]. In this case every 2-shadow K_ξ can be translated inside the corresponding shadow L_ξ (by Theorem 1.3), while the (Euclidean) volumes of L and K satisfy $V(L) < V(K)$. This implies that K and L are not homothetic. Now dilate L sufficiently so that $V(L) = V(K)$. The triangle covering condition is preserved, but K and L are not translates.

2. Most objects may be hidden without being covered

We have shown that, if the d-shadows of a compact convex set L cover the d-shadows of a polytope Q having at most $d+1$ vertices, then L contains a translate of Q. What if Q has more vertices? What if Q is replaced by a more general compact convex set K? It turns out that adding one additional vertex changes the story.

Consider, for example, a regular tetrahedron Δ in $\mathbb{R}^3$. Let Q be a planar quadrilateral with one vertex from the relative interior of each facet of Δ. Since Q does not meet any edge of Δ, every 2-shadow of Q has a translate inside the *interior* of the corresponding 2-shadow of Δ. By a standard compactness argument, there is an $\epsilon > 1$ such that every 2-shadow of ϵQ can be translated inside the corresponding 2-shadow of Δ. But Q already meets every facet of Δ, so the simplex Δ cannot contain any translate of ϵQ.

More generally, we will show that if $K \in \mathscr{K}_n$ has more than $d + 1$ exposed points, then there exists $L \in \mathscr{K}_n$ whose d-shadows contain translates of the corresponding d-shadows of K, while L does not contain a translate of K.

Lemma 2.1. *Let Δ be an n-simplex, and let $K \subseteq \Delta$ be a compact convex set. Suppose that $K \cap F = \emptyset$ for every face F of Δ such that* $\dim(F) \leq n-2$. *Then*

- **(i)** *For each $u \in \mathbb{S}^{n-1}$, the projection K_u can be translated inside the interior of Δ_u.*
- **(ii)** *There exists $\epsilon > 1$ such that, for each u, the projection Δ_u contains a translate of ϵK_u.*

Note that the value ϵ in **(ii)** is independent of the direction u.

Proof. Let $u \in \mathbb{S}^{n-1}$. The projection Δ_u is either an $(n-1)$-simplex or a polytope in $u^\perp$ having $n+1$ vertices. Denote by $\pi_u : \mathbb{R}^n \longrightarrow u^\perp$ the orthogonal projection map onto the subspace $u^\perp$.

If $\pi_u(\Delta)$ has $n+1$ vertices, then π_u maps the relative interior of each facet of Δ into the interior of Δ_u, so that K_u lies in the interior of Δ_u.

If $\pi_u(\Delta)$ is an $(n-1)$-simplex, then at least one facet of the simplex Δ_u is the bijective image of an $(n-2)$-face of Δ under π_u. Therefore $K_u \subseteq \Delta_u$ is disjoint from that facet of Δ_u, so that a translate of K_u lies in the interior of Δ_u. This proves **(i)**.

Since the interior of each Δ_u contains a translate of K_u, there exists $\epsilon_u > 1$ such that $\epsilon_u K$ can be translated inside Δ_u. Let $\epsilon = \inf_u \epsilon_u$, and let $\{u_i\}$ be a sequence of unit vectors such that $\epsilon_i = \epsilon_{u_i}$ converge to ϵ. Since the unit sphere is compact, we can pass to a subsequence as needed, and assume without loss of generality that $u_i \to v$ for some unit vector v.

Since $\epsilon_v > 1$, we can translate K and Δ so that $o \in K_v \subseteq \Delta_v$, where the origin o now lies in the interior of Δ. If $\alpha = \frac{1+\epsilon_v}{2}$, then αK_v lies in the relative interior of Δ_v, so that their support functions satisfy $\alpha h_K(x) < h_\Delta(x)$ for all unit vectors $x \in v^\perp$. Since support functions are uniformly continuous on the unit sphere, and since $u_i \to v$, we have $\alpha h_K(x) < h_\Delta(x)$ for all $x \in u_i^\perp$ for i sufficiently large. This means that αK_{u_i} lies in the relative interior of Δ_{u_i} for large i, so that $\alpha < \epsilon_i$ as well. Taking limits, we have $1 < \alpha \leq \epsilon$. Since $\epsilon > 1$, the assertion **(ii)** now follows. □

A set $C \subseteq \mathbb{S}^{n-1}$ is a closed spherical convex set if C is an intersection of closed hemispheres. The polar dual C^* is defined by

$$C^* = \{u \in \mathbb{S}^{n-1} \mid u \cdot v \leq 0 \text{ for all } v \in C\}.$$

If $x \in C \cap C^*$ then $x \cdot x = 0$. This is impossible for a unit vector x, so we have $C \cap C^* = \emptyset$. Recall also that $C^{**} = C$. See, for example, [14, 16]. (Note that one can identify C with the cone obtained by taking all nonnegative linear combinations in $\mathbb{R}^n$ of points in C, taking the polar dual in this context, and then intersecting with the sphere once again.)

Lemma 2.2. *Let C be a closed spherical convex set in $\mathbb{S}^{n-1}$. Then there exists a unit vector $v \in -C \cap C^*$.*

Moreover, if C has dimension $j \geq 0$ and lies in the interior of a hemisphere, then $-C \cap C^*$ also has dimension j.

Proof. Since $C \cap C^* = \emptyset$, there is a hyperplane $H = v^\perp$ through the origin in $\mathbb{R}^n$ that separates them. Let H^+ and H^- denote the closed hemispheres bounded by $H \cap \mathbb{S}^{n-1}$, labelled so that $v \in H^+$, and so that $C \subseteq H^-$ and $C^* \subseteq H^+$.

Since $C \subseteq H^- \subseteq \{v\}^*$, we have $v \in C^*$. (Polar duality reverses inclusion relations.) Meanwhile, $C^* \subseteq H^+ = -\{v\}^* = \{-v\}^*$, so that $-v \in C^{**} = C$, and $v \in -C$. Conversely, if $v \in -C \cap C^*$ then $v^\perp$ separates C and C^*.

If C has dimension $j \geq 0$ and lies in the interior of a hemisphere, then C^* has interior, and the set $C^* \cap -C$ consists of all v such that $v^\perp$ separates C and C^*, a set of dimension j as well. □

Theorem 2.3. *If $K \in \mathscr{K}_n$ has dimension n, then there exist regular unit normal vectors $u_0, \ldots, u_n$, at distinct exposed points $x_0, \ldots, x_n$ on the boundary of K,* *such that $u_0, \dots, u_n$ are the outward unit normals vectors of some n-dimensional simplex in $\mathbb{R}^n$.*

Note that Theorem 2.3 is trivial if K is smooth and strictly convex, where each supporting hyperplane of K meets K at a single boundary point, and each boundary point has exactly one supporting hyperplane. In this case, *any* circumscribing n-simplex for K will do.

If K is a polytope, then Theorem 2.3 is again easy to prove, since each exposed point (vertex) of K has a unit outward normal cone with interior in the unit sphere, and these interiors fill the sphere except for a set of measure zero. Once again we can take any circumscribing simplex S for K, and then make small perturbations of each facet normal so the each facet of S meets a different vertex of K.

The following more technical argument verifies Theorem 2.3 for arbitrary $K \in \mathscr{K}_n$ having dimension n (i.e. having non-empty interior).

Proof of Theorem 2.3. If x lies on the boundary of K, denote by $N(K, x)$ the outward unit normal cone to K at x; that is,

$$N(K, x) = \{u \in \mathbb{S}^{n-1} \mid x \cdot u = h_K(u)\}.$$

Let u_0 be a regular unit normal at the exposed point $x_0 = K^{u_0}$. By the previous lemma, we can choose u_0 in the normal cone $N_0 = N(K, x_0)$ so that $u_0 \in N(K, x_0) \cap -N(K, x_0)^*$.

Since K has dimension n, the normal cone N_0 lies in an open hemisphere. Recall that regular unit normal vectors to K are dense in the unit sphere $\mathbb{S}^{n-1}$ (see [14, p. 77]). It follows that we can choose u_1, x_1, N_1 similarly, so that u_1 lies outside N_0 and so that $\{u_0, u_1\}$ are linearly independent. Once again N_1 lies inside an open hemisphere.

Having chosen u_i, x_i, N_i in this manner, for $i = 0, \dots, k$, where $k < n - 1$, the union $N_0 \cup \ldots \cup N_k$ cannot cover the sphere, because each is a closed subset of an open hemisphere, and the S^{n-1} is not the union of $n - 1$ open hemispheres. It follows that

$$X = \mathbb{S}^{n-1} - (N_0 \cup \ldots \cup N_k)$$

is a nonempty open subset of $\mathbb{S}^{n-1}$. Since regular unit normals to K are dense in the sphere, we can choose $u_{k+1} \in X$ so that x_{k+1} is disjoint from the previous choices of x_i, and such that $u_0, \dots, u_{k+1}$ are linearly independent.

Continuing in this manner, we obtain a linearly independent set $u_0, \dots, u_{n-1}$ of regular unit normals at distinct exposed points $x_0, \dots, x_{n-1}$ of K. Since the unit normals $u_0, \dots, u_{n-1}$ are independent, the origin o does *not* lie in their convex hull. Therefore, there exists an open hemisphere containing $u_0, \dots, u_{n-1}$, and we can take spherical convex hull of $u_0, \dots, u_{n-1}$, to be denoted C. Again, since the u_i are independent, the set C has interior. Since C is contained inside an open hemisphere, C^* also has interior. By the previous lemma, $C^* \cap -C$ is non-empty and open. By the density of regular normals, there exists regular unit normal u for K such that u lies in the interior of $C^* \cap -C$. Since u lies in the interior of C^*, each $u \cdot u_i < 0$, so that $u \notin N_i$ for any i (by our choice of each $u_i \in N_i$). It follows that $x = K^u$ is distinct from the previous exposed points $x_0, \ldots, x_{n-1}$. Moreover, since u lies in the interior of $-C$

$$-u = a_0 u_0 + \cdots + a_{n-1} u_{n-1}$$

for some $a_i > 0$, so that

$$a_0 u_0 + \cdots + a_{n-1} u_{n-1} + u = o.$$

Set $u_n = u$ and $x_n = x$. The Minkowski existence theorem [1, p. 125][14, p. 390] (or a much simpler Cramer's rule argument) yields an n-simplex with unit normals $u_0, \ldots, u_n$. Scaling this simplex to circumscribe K, each ith facet will meet the boundary of K at exactly the distinct exposed point x_i. □

Theorem 2.4. *If $K \in \mathscr{K}_n$ has at least $n+1$ exposed points, then there exists a simplex $S \in \mathscr{K}_n$ such that each projection S_u contains a translate of the projection K_u, while S does not contain a translate of K.*

Proof. If $\dim(K) = n$ then Theorem 2.4 immediately follows from Theorem 2.3 and Lemma 2.1.

If $\dim(K) = d < n$, let ξ denote the affine hull of K. By Theorem 2.3, there exists a a d-dimensional simplex $Q \subseteq \xi$ that circumscribes K in ξ and whose $d+1$ facet unit normals are regular unit normals of K. Since K has $n+1$ exposed points, there are (at least) another $n-d$ regular unit normals of K (in ξ) at these additional exposed points. After intersecting Q with supporting half-spaces (in ξ) of K relative to these additional $n-d$ normals, we obtain a polytope Q_1 in ξ whose $n+1$ facet unit normals are regular unit normals of K. Since $\dim \xi < n$, apply small perturbations of these $n+1$ facet unit normals to Q along $\xi^\perp$ to obtain facet normals of a simplex S in $\mathbb{R}^n$, whose facet normals are still regular unit normals to K in $\mathbb{R}^n$.

In either instance, we have obtained a simplex $S \supseteq K$, so that K meets the boundary of S at exactly $n+1$ points, one point from the relative interior of each facet of S. By Lemma 2.1 there exists $\epsilon > 1$ such that ϵK_u can be translated inside S_u for all u. But ϵK cannot be translated inside S, since S circumscribes K already, and $\epsilon > 1$. □

Corollary 2.5. *If $K \in \mathscr{K}_n$ and* $\dim(K) = n$*, then there exists $L \in \mathscr{K}_n$ such that each projection L_u contains a translate of the projection K_u, while L does not contain a translate of K.*

The following proposition addresses an ambiguity regarding when shadows cover inside a larger ambient space.

Proposition 2.6. *Suppose that ξ is a linear flat in $\mathbb{R}^n$. Let K and L be compact convex sets in ξ. Suppose that, for each d-subspace $\eta \subseteq \xi$, the projection L_η contains a translate of K_η. Then L_η contains a translate of K_η for every d-subspace $\eta \subseteq \mathbb{R}^n$.*

Proof. Suppose that η is a d-subspace of $\mathbb{R}^n$. Let $\hat{\eta}$ denote the orthogonal projection of η into ξ. Since $\dim(\hat{\eta}) \leq \dim(\eta) = d$, we can translate K and L inside ξ so that $K_{\hat{\eta}} \subseteq L_{\hat{\eta}}$. Let us assume this translation has taken place. Note that, for $v \in \hat{\eta}$, we now have $h_K(v) \leq h_L(v)$.

If $u \in \eta$, then express $u = u_\xi + u_{\xi^\perp}$. Since $K \subseteq \xi$,

$$h_K(u) = \max_{x \in K} x \cdot u = \max_{x \in K} x \cdot u_\xi = h_K(u_\xi),$$

and similarly for L. But since $u \in \eta$, we have $u_\xi \in \hat{\eta}$, so that

$$h_K(u) = h_K(u_\xi) \leq h_L(u_\xi) = h_L(u).$$

In other words, $K_\eta \subseteq L_\eta$. □

Theorem 2.4 can now be generalized.

Theorem 2.7. *Suppose that $d \in \{1, 2, \ldots, n-1\}$. If K has at least $d+2$ exposed points, then there exists $L \in \mathscr{K}_n$ such that the projection L_ξ contains a translate of the projection K_ξ for each d-dimensional subspace ξ, while L does not contain a translate of K.*

Proof. Note that $n > d$. If $n = d+1$ then Theorem 2.4 applies, and we are done.

Suppose that Theorem 2.7 holds when $n = d+i$ for some $i \geq 1$. If $n = d+i+1$, then there are two possible cases to consider.

First, if $\dim K = n$, then Corollary 2.5 yields $L \in \mathscr{K}_n$ such that every shadow L_u contains a translate of K_u, while L does not contain a translate of K. Since every d-subspace ξ is contained in some hyperplane $u^\perp$, it follows *a fortiori* that every d-dimensional shadow L_ξ contains a translate of K_ξ as well.

Second, if $\dim K < n$, the induction hypothesis holds in the (lower dimensional) affine hull $\mathrm{Aff}(K)$ of K. In other words, there exists a compact convex set L in $\mathrm{Aff}(K)$ such that the projection L_ξ contains a translate of the projection K_ξ for each d-dimensional subspace ξ of $\mathrm{Aff}(K)$, while L does not contain a translate of K. Since $\mathrm{Aff}(K)$ is a flat in $\mathbb{R}^n$, inclusion of L in $\mathbb{R}^n$ preserves these covering properties, by Proposition 2.6. □

Corollary 2.8. *If $\dim K = d+1$, where $d \leq n-1$, then there exists $L \in \mathscr{K}_n$ such that the projection L_ξ contains a translate of the projection K_ξ for each d-dimensional subspace ξ, while L does not contain a translate of K.*

Proof. If $\dim K = d+1$ then K must have at least $d+2$ exposed points [16, p. 89], so that Theorem 2.7 applies. □

3. Concluding remarks

Although we have restricted our covering questions to shadows given by *orthogonal* projections, the next proposition shows that the same results will apply when more general (possibly oblique) linear projections are admitted.

Proposition 3.1. *Let $K, L \in \mathscr{K}_n$. Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a nonsingular linear transformation. Then L_u contains a translate of K_u for all unit directions u if and only if $(\psi L)_u$ contains a translate of $(\psi K)_u$ for all u.*

Proof. For $S \subseteq \mathbb{R}^n$ and a nonzero vector u, let $\mathcal{L}_S(u)$ denote the set of straight lines in $\mathbb{R}^n$ parallel to u and meeting the set S. The projection L_u contains a translate K_u for each unit vector u if and only if, for each u, there exists v_u such that

$$\mathcal{L}_{K+v_u}(u) \subseteq \mathcal{L}_L(u). \tag{3}$$

But $\mathcal{L}_{K+v_u}(u) = \mathcal{L}_K(u) + v_u$ and $\psi\mathcal{L}_K(u) = \mathcal{L}_{\psi K}(\psi u)$. It follows that (3) holds if and only if $\mathcal{L}_K(u) + v_u \subseteq \mathcal{L}_L(u)$, which in turn holds if and only if

$$\mathcal{L}_{\psi K}(\psi u) + \psi v_u \subseteq \mathcal{L}_{\psi L}(\psi u) \quad \text{for all unit } u.$$

Set

$$\tilde{u} = \frac{\psi u}{|\psi u|} \quad \text{and} \quad \tilde{v} = \psi v_u.$$

The relation (3) now holds if and only if, for all $\tilde{u}$, there exists $\tilde{v}$ such that

$$\mathcal{L}_{\psi K}(\tilde{u}) + \tilde{v} \subseteq \mathcal{L}_{\psi L}(\tilde{u}),$$

which holds if and only if $(\psi L)_{\tilde{u}}$ contains a translate of $(\psi K)_{\tilde{u}}$ for all $\tilde{u}$. □

In this note we have addressed the existence of a compact convex set L, whose shadows can cover those of a given set K, without containing a translate of K itself. A reverse question is addressed in [9]: Given a body L, does there necessarily exist K so that the shadows of L can cover those of K, while L does not contain a translate of K? A body L is called *d-decomposable* if L is a *direct* Minkowski sum (affine Cartesian product) of two or more convex bodies each of dimension at most d. A body L is called *d-reliable* if, whenever each d-shadow of K can be translated inside the corresponding shadow of L, it follows that K can itself be translated inside L. In [9] it is shown that d-decomposability implies d-reliability, although the converse is (usually) false. The results in [8, 9], along with those of the present article, motivate the following related open questions:

I. Under what symmetry (or other) conditions on a compact convex set L in $\mathbb{R}^n$ is d-reliability equivalent to d-decomposability, for $d > 2$?

In [9] it is shown that 1-reliability is equivalent to 1-decomposability. That is, only parallelotopes are 1-reliable. It is also shown that a centrally symmetric compact convex set is 2-reliable if and only if it is 2-decomposable. However, this equivalence fails for bodies that are not centrally symmetric.

Denote the n-dimensional (Euclidean) volume of $L \in \mathscr{K}_n$ by $V_n(L)$.

II. Let $K, L \in \mathscr{K}_n$ such that $V_n(L) > 0$, and let $1 \leq d \leq n-1$. Suppose that the orthogonal projection L_ξ contains a translate of the projection K_ξ for all d-subspaces ξ of $\mathbb{R}^n$.

What is the best upper bound for the ratio $\frac{V_n(K)}{V_n(L)}$?

In [8] it shown that $V_n(K)$ may exceed $V_n(L)$, although $V_n(K) \leq nV_n(L)$. This crude bound can surely be improved.

III. Let $K, L \in \mathscr{K}_n$, and let $1 \leq d \leq n-1$. Suppose that, for each d-subspace ξ of $\mathbb{R}^n$, the orthogonal projection K_ξ of K can be moved inside L_ξ by some *rigid motion* (i.e. a combination of translations, rotations, and reflections).

Under what simple (easy to state, easy to verify) additional conditions does it follow that K can be moved inside L by a rigid motion?

Because of the non-commutative nature of rigid motions (as compared to translations), covering via rigid motions may be more difficult to characterize than the case in which only translation is allowed.

References

1. T. Bonnesen and W. Fenchel, *Theory of Convex Bodies*, BCS Associates, Moscow, Idaho, 1987.
2. R. J. Gardner, *Geometric Tomography (2nd Ed.)*, Cambridge University Press, New York, 2006.
3. R. J. Gardner and A. Volčič, *Convex bodies with similar projections*, Proc. Amer. Math. Soc. **121** (1994), 563–568.
4. H. Groemer, *Ein Satz über konvexe Körper und deren Projektionen*, Portugal. Math. **21** (1962), 41–43.
5. H. Hadwiger, *Gegenseitige Bedeckbarkeit zweier Eibereiche und Isoperimetrie*, Vierteljschr. Naturforsch. Gesellsch. Zürich **86** (1941), 152–156.
6. ———, *Überdeckung ebener Bereiche durch Kreise und Quadrate*, Comment. Math. Helv. **13** (1941), 195–200.
7. ———, *Seitenrisse konvexer Körper und Homothetie*, Elem. Math. **18** (1963), 97–98.
8. D. Klain, *Covering shadows with a smaller volume*, arXiv:0804.2718v3 (2009).
9. ———, *If you can hide behind it, can you hide inside it?*, preprint (2009).
10. D. Klain and G.-C. Rota, *Introduction to Geometric Probability*, Cambridge University Press, New York, 1997.
11. E. Lutwak, *Containment and circumscribing simplices*, Discrete Comput. Geom. **19** (1998), 229–235.
12. C. A. Rogers, *Sections and projections of convex bodies*, Portugal. Math. **24** (1965), 99–103.

13. L. A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley, Reading, MA, 1976.
14. R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, New York, 1993.
15. V. Soltan, *Convex sets with homothetic projections*, arXiv:0903.2836v1 (2009).
16. R. Webster, *Convexity*, Oxford University Press, New York, 1994.
17. J. Zhou, *The sufficient condition for a convex body to contain another in* $\mathbb{R}^4$, Proc. Amer. Math. Soc. **121** (1994), 907–913.
18. ———, *Sufficient conditions for one domain to contain another in a space of constant curvature*, Proc. Amer. Math. Soc. **126** (1998), 2797–2803.